## GEORGE ABRAM MILLER

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George Abram Miller was born on a farm near Lynville, Pennsylvania, on July 31, 1863, and died at Urbana, Illinois, on February 10, 1951. He had long been established in the front rank of the mathematicians of the world when he assisted at the founding of the Mathematical Association of America on December 30, 1915. He was one of the two vice-presidents in 1916, and was president in 1921. He was made an honorary life member in 1937. His connection with the MONTHLY began in its earliest days and the earliest days of his mathematical career. In volumes 2 and 3, in 1895 and 1896, he published in a series of fourteen installments some forty pages of exposition of substitution groups. This set of papers constitutes the simplest and most complete account of the substitution groups of low degree in the literature. Thereafter he contributed seventy papers to the MONTHLY, the last in 1933 when he was seventy years old. In 1909 the University of Illinois joined the University of Chicago in support of the MONTHLY, and Miller became one of the editors; he remained on the editorial board until 1915. The last issue of the MONTHLY in 1909 carried the announcement of Editor Miller's marriage to Miss Cassandra Boggs of Urbana; she was an understanding and cherished companion for forty years.

His family spoke German at home; the schools were conducted in English. He became a teacher at seventeen to earn money to prepare for and to attend college. He attended Franklin and Marshall Academy, a subdivision of the college of that name at Lancaster, in 1882–1883, and Muhlenberg College from 1884 to 1887. He was graduated from Muhlenberg with honorable mention, ranking third in his class. The next year he was Principal in the public schools of Greeley, Kansas, and then he became Professor of Mathematics at Eureka College at Eureka, Illinois. His preparation for this position was meager, but the requirements were modest. The curriculum at Muhlenberg offered calculus in the third year and astronomy and meteorology in the fourth; Eureka required a knowledge of calculus for the Bachelor of Arts degree. After the first year he spent part of the summer of 1889 at the Johns Hopkins University, and he spent the summer of 1890 at the University of Michigan; neither university was in session during those summers. He was made a Master of Arts by Muhlenberg in 1890, but the degree was offered to every Bachelor of Arts of three years' standing who was of good moral character and had been engaged in liberal or professional pursuits. In the meantime he had started the collection of a mathematics library at Eureka and in 1890 he offered a set of advanced courses leading to the degree of Doctor of Philosophy on the completion of two years of satisfactory work. In the year 1891–1892 he was enrolled by correspondence as a graduate student in Cumberland University, where he was granted the degree of Doctor of Philosophy in 1892, thus becoming the only Ph.D. on the staff at Eureka. In 1890 the staff of the University of Illinois had four Ph.D.'s and one D.Sc.; Eureka had about 300 students and Illinois 351.

In 1893 Miller went to the University of Michigan as an Instructor. With the encouragement of F. N. Cole he started upon the study of finite groups which he followed for the rest of his life. Cole had spent two years in Germany where he had studied under Lie and Klein; he had returned to receive his doctor's degree at Harvard; he had lectured on groups at Harvard in 1886, and those lectures have been said to have inaugurated a new era in instruction in advanced mathematics there. Miller's first two papers were published in volume 3 (1894) of the Bulletin of the New York Mathematical Society. They were the result of the redetermination of the substitution groups of degrees eight and nine. Cayley had given a list of the former in 1892 and Cole had corrected it; Cole had listed the groups of degree nine. Miller added two groups to each list, bringing the totals to 200 and 258, respectively, and in the latter case he corrected Jordan's list of the primitive groups of degree nine published in 1872. In the same year he published his own list of 994 intransitive groups of degree ten. These lists have not been questioned since that time.

This successful measuring of his powers against Cole and Jordan and Cayley was the encouragement he needed. In 1895–1897 he attended the lectures of Lie in Leipzig and Jordan in Paris, but his time was spent very much as it would have been spent if he had remained at Ann Arbor; he had published 27 papers by the end of the year of his return. By 1900 he was acknowledged to be in the front rank of students of groups in the country, and by 1910 he was unsurpassed in the world.

His publications on abstract groups began in 1896. In his first paper he determined for the first time the 15 groups of order 24 and the 51 groups of order 32. In 1930 he determined the 294 groups of order 64. The fact that in the meantime there had been only one attempt, and that unsuccessful, to determine the groups of order 64 gives an indication of the difficulty of the problem. Throughout his life he worked directly on and very close to the problem of the determination of the groups of order $n$. Very few of the results which he obtained have been improved upon.

After Miller's return from Europe in 1897, he spent four years at Cornell, five years at Leland Stanford, and then nearly forty-five years at Illinois. He was a member of the mathematical societies of England, Germany, Spain, India, and America. He was elected to high office in the American Mathematical Society and the Association for the Advancement of Science, and he held membership in the National Academy of Sciences and the American Academy of Arts and Sciences. He was made an honorary Doctor of Letters by Muhlenburg in 1936.

After his retirement in 1931 he followed the program he had been following before. He came to his office morning, afternoon, and evening regardless of the calendar, the weather, or those minor ills that plague us all and occasionally keep some of us at home. In 1935 he published 18 papers and in 1944, when he was eighty-one, he published ten. His last publications were two short papers in 1947. He came regularly to his office until Christmas, 1950.

He published two books, *Determinants* (1892), and *Historical Introduction to Mathematical Literature* (1916). He wrote parts of two more, *The Algebraic Equation* in *Monographs on Topics of Modern Mathematics* (1911) and the first part of *Finite Groups* (1916) by Miller, Blichfeldt, and Dickson. He published some 820 papers in the educational, scientific, and mathematical journals of eleven countries. His non-technical papers included expositions of groups for non-specialists, expositions of mathematics for non-mathematicians, and studies on the history of mathematics.

His 450 (approximately) technical papers on groups constitute a permanent addition to the knowledge of finite groups. A knowledge of the substitution groups of low degrees and the abstract groups of low orders has a value in situations far removed from the obvious ones. He was the first to prove many of the things that every student of the subject uses. In some directions he carried his investigations far enough to show that lines of development which looked promising are not feasible.

G. A. Miller was a man of power which he directed to worthy ends; he hastened the development of his subject; and he added largely to the prestige of American mathematics.

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## THE STORY OF TANGENTS

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**1. The Greeks.** Whoever has given the least thought to the subject of plane curves has given some consideration to tangents. But what are tangents? To the uninstructed, a plane curve, not a straight line, is the path traced by a moving point whose motion changes its direction at each instant. Had the point at any instant decided not to change the direction of motion, the line through that point in the direction of instantaneous motion would be the tangent. All of this is perfectly clear to any observant and unsophisticated person who has not endeavored to go below the surface, it is only when some one suggests awkward questions like "What do you mean by direction?" "What is meant by instantaneous motion?" that difficulties begin to appear.

Perhaps a beginner starts with a static approach. Here is an arc of a curve. We take a point on the arc and draw through it a line which meets the curve there and nowhere else in the vicinity. This line we may call the tangent and prove, if we can, that it is unique. In the case of the circle it is perpendicular to the radius to that point. But suppose we start with the conic sections and ask whether in the case of a hyperbola a line parallel to an asymptote, which certainly does not meet the curve elsewhere, is really a tangent. There is clearly no

suggestion of touching in such a case. It is clear that we can not be too naïve in answering such questions, it is the purpose of the present article to tell the story of how men have attempted to answer them [1].

I can not make out that before Euclid anyone was much concerned with tangents. I turn therefore to Euclid Book III, Definition I [2].

*A straight line is said to touch a circle which meeting the circle and, being produced, does not cut the circle.*

I am no Greek scholar so I follow what Heath says on the following page of his commentary. He draws a distinction between ἅπτεσθαι which means to meet, and ἐφάπτεσθαι which means to touch. He suggests that this distinction was common with the Greeks, although he cites exceptions one way or the other, one being in the work of Archimedes, though the exact reference to the passage is not given. In the work of Archimedes on spirals, to which I shall presently return, the word used is επιψαυη which Heath also translates "touch" [3]. I think there is no difficulty in seeing what Euclid means. He expands the idea further, for in Euclid III (16) we have

"*The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line can not be interposed.*" The tangent meets the circle once only and lies wholly outside, and no other line through the point of contact has this property.

When we come to the conic sections, we learn from Apollonius I, (17) and (32) "*If a straight line be drawn through the extremity of the diameter of any conic parallel to the ordinates to that diameter, the straight line will touch the conic, and no other straight line can fall between it and the conic.*" This theorem can not, however, have been original with Apollonius, for he speaks of his first four books as having been worked out by his predecessors. Archimedes in the first proposition on the quadrature of the parabola remarks, "These propositions are proved in the elements of the conics," referring, probably, to the works of Euclid and Aristaeus. I think it safe to say that Euclid knew the Apollonian theorems which I have given.

It is time to turn to Archimedes, especially to his work on spirals. An Archimedian spiral is the curve traced by a point advancing uniformly on a line which turns uniformly about a fixed point. Here we run across the very curious fact that although he speaks frequently of tangents to spirals, he does not define them. I think it is evident that he means by a tangent a straight line which meets the spiral at a point but does not cross it there; a tangent meets the curve at one point but nowhere else in the vicinity. He could doubtless have proved that no other line lies between the tangent and the curve.

Archimedes shows next that as the radius vector and angle both increase uniformly, presumably with the time, if a series of angles form an arithmetical progression, the same is true of the corresponding radii vectores. If thus we take the radii vectores to two points of the same turn of the spiral, the radius vector which bisects the angle between them is equal in length to one half their sum.

This leads to the rather curious Proposition 13, *If a straight line touch a spiral, it will touch it once only*. This means at only one point of that turn, as use is made of the preceding theorem about arithmetical series. Let $O$ be the origin, and suppose a certain tangent touches the curve at both $P$ and $Q$. Let $OR$ be the radius vector which bisects the angle. Then $OP+OQ=2OR$. But we can show, though Archimedes does not do so, the distance to $PQ$ along the bisector is less than this, so there is a point on $PQ$ inside the curve, but this is contrary to the hypothesis that the line $PQ$ was a tangent. As a matter of fact the line $PQ$ meets the curve an infinite number of times, so there are plenty of points on both sides of it but not in the $POQ$ sector. The whole treatment seems to me much looser than we have a right to expect from this wonderful geometer.

Archimedes' great accomplishment is to give an actual construction for the tangent. This is done by giving the polar sub-tangent, that is to say, the distance along the perpendicular at the origin to the radius vector from the origin to the tangent, the distance which we should write $r^2d\theta/dr$. Here he uses the method of exhaustion and the method of verging, which consists in placing a line segment of given length with its ends on given curves while its line passes through given points. His fundamental theorem is that if $P$ be a point of the $n$th turn and $OP=r$, the polar sub-tangent will be $a=2\pi(n-1)r+\text{arc } AP$ where $A$ is the starting point. The proof I find difficult to follow. It consists in showing that if the theorem were not true we could find a point outside the spiral which should be analytically inside. I can not see where the actual tangency comes in, the proof merely shows that in any other case an impossible construction would result. An analytical expansion of the method will be found in [4].

How did Archimedes discover this result? We do not know, but we can make a shrewd guess. In his *Method*, Archimedes shows how he was first led to theorems about areas and volumes, which he proved rigorously by the method of exhaustion, by cutting his figures into slices and then comparing their turning moments in different positions. We have here the concept of infinitely thin slices. Is it not likely that he also had the idea of a tangent as the line of an infinitely short chord, a perfectly familar concept to the mathematicians of the seventeenth century. If, then, $a$ be the polar sub-tangent corresponding to the infinitely small advances $dr$, $rd\theta$, we have by similar triangles

$$\frac{a}{r}=\frac{rd\theta}{dr},$$

but in the case of the spiral of Archimedes

$$r=k\theta, \qquad a=r\theta$$

and this is Archimedes rule.

**2. Fermat and Descartes.** The idea of a tangent as the limiting position of a secant when two intersections with the curve tend to fall together was slow in attaining mathematical acceptance. The fact is that the idea of a limiting posi-

tion of any sort came to birth slowly and painfully. But the idea of two intersections falling together was well understood before the middle of the seventeenth century. I think that here we should lay great stress on the work of Fermat, who certainly had it clearly in mind even though certain modern writers are very insistent that he did not understand at all the limit idea.

The ideas of Fermat were first set forth in 1629 in a letter to a certain M. Despagnet [5]. The method of finding tangents seems to have been a bi-product of his method of finding maxima and minima. Whether this is really an anticipation of the method of differentiation is disputable, an elaborate discussion by Wieleitner is found in [6]. The best explanation of what is found occurs much later in a letter of 1643 to Brulart de St. Martin [7].

Suppose that we seek a maximum or a minimum for a function $f$. This will appear on the graph as the top or bottom of an arch at a specific point $A$. The functions $f(A+E)$ and $f(A-E)$ will both be greater than or less than $f(A)$. Suppose, then, that we write

$$f(A \pm E) = f(A) \pm Ef'(A) + \frac{E^2}{2} f''(A) + \cdots .$$

The function $f$ is supposed to be a polynomial, and so the higher powers of $E$ may be neglected. We have then

$$f(A + E) = f(A - E); \qquad f'(A) = 0.$$

The particular example that he takes is $f(A) \equiv BA^2 - A^3$. A similar process is applied to the problem of finding the tangent to a parabola; finding the tangent does not mean finding its equation but finding the sub-tangent [8]. He ends the article with the claim "Nec unquam fallit methodus." A somewhat different explanation is found on page 147 of Vol. I of [5], and Wieleitner in [6] makes much of the difference; neither account strikes me as particularly convincing.

Fermat makes what I think is a much better use of his method on page 158 ff. of Vol. I of [5]. Let the equation of the curve be $F(x, y) = 0$. We seek the subtangent at the point $(x, y)$. A very near point shall be $(x+e)$. The ordinate up to the tangent is found by similar triangles to be $y(1+e/a)$ and this we treat as if it were also on the curve, so that

$$F(x, y) = F\left(x + e, y\left(1 + \frac{e}{a}\right)\right) = 0.$$

He takes the "cissoid" of Diocles and the "conchoid" of Nicomedes, finding $a$ by this limiting process. He points out, incidentally, that to find an inflexion, we must find a maximum or minimum of the angle which a tangent makes with a given direction, and this means finding a maximum or a minimum of its cotangent, and that means to maximize or minimize $a/y$, where $a$ is the subtangent.

A good example of Fermat's method is found in the case of the "folium" of Descartes, as here we do not have $y$ as an explicit function of $x$:

$$x^3 + y^3 = nxy,$$

$$(x+e)^3 + y^3\left(1+\frac{e}{a}\right)^3 - ny(x+e)\left(1+\frac{e}{a}\right) = 0,$$

$$e\left(3x^2 + \frac{3y^3}{a} - \frac{nxy}{a} - ny\right) + e^2\left(3x + \frac{3y^3}{a^2} - \frac{ny}{a}\right) + e^3\left(1 + \frac{y}{a^2}\right) = 0.$$

This holds for all values of $e$. We therefore divide $e$ out, and for the tangent assume $e=0$, then

$$a = -\frac{3y^3 - nxy}{3x^2 - ny}.$$

I will point out that this amounts to putting $a = -yF_y/F_x$ but this general formula did not appear before the work of de Sluse, which we shall see later.

An interesting example is Fermat's oval, which may be written

$$y = K\sqrt{\cos\frac{x}{b}}.$$

Here we introduce an auxiliary circle of radius $b$, and $x$ becomes the length of an arc on this circle. The process of assuming the identity of two near points is used with regard to this, rather than the original curve. Immediately following the foregoing are examples of finding the tangent to one curve from the tangent to another, a process also followed by Barrow. Corresponding abscissas are equal, and the ordinate to one equals the arc-length to the other. He shows later that the slope of one is equal to the secant of the slope-angle of the other. He solves the problem in the case of the parabola twice, once by his own methods, once by what he calls the "Ancients" method where a tangent is defined as a line meeting a curve but once in a certain region. A preliminary theorem tells us that if we take a point of an arc whose tangent turns continually in one way, and proceed to a nearby ordinate, the distance along the tangent is less than that along the curve, if this be the direction of decreasing ordinates, but greater if it be the direction of increasing ordinates. The proof consists in applying the principle that an arc is longer than its chord, but less than the sum of two tangents from an external point to its ends. This is preliminary to an elaborate study of the lengths of curves.

I can not leave Fermat without expressing admiration for his method, which is essentially that of the infinitesimal calculus, even if he did not see all that was involved. He had, I think, a better grasp of the essential principles than his contemporaries, and was certainly early, perhaps the earliest, in the field.

It is time to pass to Fermat's great rival in this matter, René Descartes. He first attacked the problem of tangents in 1637, which was later than Fermat's

letter to Despagnet, but before he had heard directly on the subject from Fermat. It is very curious that Descartes began by seeking to draw a normal to a curve, that is to say, a perpendicular to a tangent at its point of contact [10]. Which straight line cuts a curve at right angles at a given point, or, as he says, cuts the "contingent" at right angles? If $(x, y)$ be the given point, and if $(x, 0)$ be where the normal meets the $X$-axis, it is the center of a circle, two of whose intersections with the curve fall together. Thus, if we write,

$$(X - x_1)^2 + Y^2 = (x - x_1)^2 + y^2, \qquad f(X, Y) = 0,$$

and eliminate $Y$, a necessary condition for a normal is that two of the roots of this equation in $X$ should fall together. If we take the case of the parabola,

$$Y^2 = 2mX,$$
$$(X - x_1)^2 + 2mX = (x - x_1)^2 + 2mx,$$
$$X^2 + 2X(m - x_1) = x^2 + 2x(m - x_1).$$

The two roots will be equal if $x_1 = m + x$.

In the examples which Descartes worked out, he did not have available a general method of handling the case of equal roots. He wrote

$$\Phi(X) \equiv (x - x_1)^2(c_0X^m + c_1X^{m-1} + \cdots + c_m)$$

and identified the coefficients on the two sides. The general rule was first worked out by Hudde in 1683 [11].

In 1638, when Descartes first heard of Fermat's method for maxima and minima, he was not a little stirred up, and expressed his disgust in letters to Mersenne, Hardy and others. He disliked especially the quantity $e$, which was divided out, because it was not equal to 0, and then equated to 0. He attacked Fermat's method of tangents as though it involved considerations of maxima and minima. In this Descartes was wrong. Fermat did not say that finding a tangent was a maximum-minimum problem, but that the methods developed for one case, also fitted the other, and this Descartes finally saw. He put his own method in a letter to Hardy of June 1638 [12].

Let

$$y^3 = lx.$$

Let us find two points $(x, y)$, $(x_1, y_1)$ so that $y_1 = ky$.

$$x_1 = x + e, \qquad y_1 = y\left(1 + \frac{e}{a}\right),$$

$$\frac{y^3}{x} = \frac{y^3\left(1 + \frac{e}{a}\right)^3}{x + e} = \frac{k^3y^3}{x + e},$$

$$a^3 = 3a^2x + 3aex + e^2x.$$

But when $k=1$, $e=0$ and $a=3x$; this does not seem to me essentially different from Fermat's method, even in the reasoning. Nor does it seem to justify the long correspondence involving Descartes, Fermat, Mersenne, Hardy and others. Descartes had, however, a third method, applied at least to the cycloid. The problem of finding a tangent to this curve had occupied various geometers. Fermat had solved it, rather clumsily. Roberval had found a solution which I shall return to later, but Descartes gave an absurdly simple solution which is as follows. Suppose [13] that a polygon rolls along a straight line, a chosen vertex will trace a succession of circular arcs whose centers are successive vertices which lie on the line. The line from the tracing vertex to the corresponding fixed vertex on the line will be perpendicular to the tangent to the arc. Now, if we consider a circle, as presently became popular, as a regular polygon of an infinite number of sides, we see that the instantaneous center is the point of contact with the given line, and the line thence to the tracing point is the normal to the cycloid. The tangent will thus always go from the point of contact to the highest point of the rolling curve.

Descartes puts the matter somewhat differently. He draws through the point of contact a line parallel to the fixed line to meet the rolling circle when it has rolled one half the distance, or when the tracing point has reached its highest position. The line from the intersection to the point of contact of the new circle is parallel to the normal sought. For any rolling curve, the tangent is perpendicular to the line from the point of contact to the instantaneous center.

**3. Roberval and Torricelli.** In discussing the work of Fermat and Descartes, we have been to a certain degree running around in circles, explaining more or less similar methods of handling infinitesimals. It was the era in which the infinitesimal calculus was struggling towards birth. Let us take a vacation from this sport by looking at a totally different approach to the subject of tangents by two more or less rival geometers whose relative merits I shall certainly not try to evaluate. There is an elaborate weighing of their comparative worth in a recent work by Evelyn Walker [22]. I, therefore, take up first Giles Persone de Roberval. He seems to have first been occupied with the classical question of the tangent to the cycloid, a curve called to his attention by Mersenne, but the best exposition of his work with tangents is found in [23].

Roberval's idea was simplicity itself. A curve is traced by a moving point; the tangent anywhere is the line of instantaneous motion of that point. The real philosophical difficulty, to define what is meant by instantaneous motion, was veiled in the future, to bedevil those of his successors who occupied themselves with the foundations of the calculus. To the unspoiled eye of common sense there was no difficulty. Here is the Axiome or principe d'invention that he gives on pp. 24 and 25c of [23]. *La direction du mouvement d'un point qui décrit une ligne courbe, est la touchante de la ligne courbe en chaque position de ce point là.* This seems to be more or less tautologous, a tangent is a line which touches, but he goes on at once to explain himself. *Par les propriétez spécifiques, (qui vous seront*

*données) examinez les divers mouvements qu'a le point à l'endroit ou vous voulez mener la touchante; de tous ces mouvements composez en un seul et tirez la ligne de direction du mouvement composé, vous aurez la touchante de la courbe.*

The meaning is this. Determine two measurements which connect the moving point with two fixed elements. Determine the vector velocities of the changes of these two. Their vector sum will give the instantaneous velocity. Gomes de Carvalho points out on page 53 of [1] that Roberval is rather cavalier in his reasoning about infinitesimal triangles; the parallelogram whose sides are $dx/dt$, $dy/dt$ is not the same as that whose sides are $dr/dt$, $rd\theta/dt$. However, in each case, the diagonal gives the direction of instantaneous motion.

Let us take some examples of Roberval's method. First, we take the parabola. The two motions are away from the focus and away from the directrix. As these two distances are always equal, the instantaneous velocities are equal. Hence the tangent makes equal angles with the axis on the focal radius. He shows carefully that this is the result given by Apollonius.

The central conics are handled in analogous fashion. We have distances from the two foci whose sum or difference is constant. Hence the differences or sums of the instantaneous velocities are constant and so the tangent makes equal angles with the focal radii, or with one radius and the other produced.

Roberval next considers the family of conchoids. Take the conchoid of Nicomedes. Lines radiate from a fixed origin to meet a directrix, a fixed line not through the origin. Each radiating line is produced a fixed distance beyond the directrix. The two motions are a radial one away from the origin and a circular one about the origin. The distance out from the directrix to the curve is independent of the choice of radius vector. If the chosen angle $\theta$ gives us $\rho$ for the directrix, and $r$ for the curve, the corresponding rotational velocities are $rd\theta/dt$ and $\rho d\theta/dt$; the stretching velocities are $dr/dt$, $d\rho/dt$. Hence the tangent of the angle which the tangent to the conchoid makes with the radius vector is $\rho/r$ times the tangent of the angle made with the directrix. The other conchoids come from another choice of directrices.

Roberval gives a simple enough construction for the tangent to the spiral if we assume, as does the Master, that we can draw a straight line equal in length to the circumference of a given circle. He becomes, however, rather deeply involved when he comes to finding the tangents to the quadratrix, or the cissoid. I confess that his analysis of the infinitesimal motions is not convincing to me. He has much better success when he comes to the cycloid. He even allows his rolling wheel to slip a bit, so that the length of the track covered in one complete turn is not necessarily equal to the circumference of the wheel. Assuming that both the sliding and turning motions are uniform, we have merely to draw, through a point on the curve, vectors parallel to the track and tangent to the wheel proportional to the distance slid and the distance turned and find their sum. In reading [23] it is easy to forget that he allows for slipping and one wonders why he does not give the simpler construction of Descartes.

Anything one says about Roberval brings to mind the name of the rival

inventor of the method of determining tangents by means of instantaneous velocities, Evangelista Torricelli. In 1644, he published his *Opera Geometrica* where, in the second section of Part 1 entitled "De motu gravium," we find expressed the technique which I will now describe. It will be found in [24]. He starts with some propositions of Galileo about falling bodies. Suppose that we start with a weighted point that falls a certain distance, then is shot off at right angles and thereafter is also allowed to fall naturally. The path will then be a parabola, for the distance slid will be proportional to the time, and the distance fallen to the square of the time. The falling velocity will also be proportional to the time, and, consequently, to the distance slid. If the parabola be $x^2=2my$ and we take the sliding velocity as the constant $x$, the ratio of dropping to sliding velocity, which will give the direction of the tangent, will be $x/m$.

We thus get Torricelli's construction for the tangent. We connect the point of contact with the reflection in the vertex of the foot of the ordinate. He adds on page 123, "Haec demonstratio peculiaris est pro parabola, sed universalem habemus pro qualibet sectione conica, consideratis aequalibus velocitatibus unius puncti, quod aequaliter movetur in utraque linea quae ex focis procedit." This is certainly very suggestive of Roberval's procedure for the central conics. He goes on to state, "Eadem ratione Demonstratur Propositio 18 de lineis spiralibus Archimedis unica brevique demonstratione, . . . Immo et hac ratione etiam unico Theoremata tangentes quarundam curvarum, inter quas, omnium linearum Cycloidalum."

Torricelli claims that he has a general method applicable not only to the parabola, but the central conics, the Archimedian spiral, and all cycloids. What is his general method? Presumably it is the composition of velocities, but he carries it out only in the one case. I have an unpleasant feeling that those who have written on the subject have not bothered to think the matter through. Jacobi writes in [25], on page 268, a direct transcript of the original with the statement "progressivi impetus ad lateralem ratio ut *ad* ad *bf* per praecedentem Propositionem," and that's all. Walker [22], page 138, states "The ratio of the progressive impetus to the lateral is as $AD:BF$. These are the ordinates of the given point and the focus, with no hint of where he gets this important fact. "Impetus" does not mean acceleration, but instantaneous velocity.

There has arisen a good deal of discussion among historians of mathematics as to which of the geometers, Roberval or Torricelli, first thought of determining tangents by instantaneous velocities. There is an elaborate discussion of the subject with dates in [22], [23], and [24]. I will not go further into the matter, but I should like to insist on the originality of the method. It is a great step forward, to pass from considering a tangent as a line meeting a curve but once, at least in a small region, to that of treating it as the limit of a secant whose intersections fall together. It was equally bold to consider it as the line of instantaneous advance.

**4. DeSluse and Barrow.** I return to the general line originated by Fermat and Descartes. An admirable extension of this was first put into words by

René François Walter, Baron deSluse. Suppose we have a curve whose equation is $f(x, y)=0$, the function being a polynomial. Reject all constant terms. Let all terms in $y$ be placed on the right with signs changed, and let each term be multiplied by the exponent of $y$. Let each term in $x$ be placed on the left, and multiplied by the exponent of $x$, and let one $x$ in each term be replaced by $a$. If a term involve both $x$ and $y$, it should appear with the proper sign on both sides and handled appropriately. We then solve for $a$, the subtangent [14] this will give

$$a = -\frac{yf_y}{f_x}.$$

The question arises at once, where did deSluse get this method? One's first idea is to credit it to Newton who had essentially the same technique, but I think the discovery must have been independent. LePaige, in the carefully written article [15], says that in 1652 deSluse had some sort of a method of drawing tangents. We find him in 1658 writing to Pascal [16], after explaining his method of drawing tangents to certain curves called "perles," "Cette méthode est tirée d'une plus universelle laquelle comprend toutes les ellipsoides, mesme avec peu de changement, les paraboloides et les hyperboloides." Newton did not begin to think about the method of fluxions before 1665. He published nothing on the subject before his paper of 1669, dictated to Collins, "De analysi per aequationes numero terminorum imfinitos." In 1673, he finally gave priority to deSluse [17].

But how did deSluse happen to hit on this technique? LePaige suggests that it comes from the formula for expanding $(x^2-y^2)/(x-y)$. I am afraid I do not see the connection. It seems to me more likely that he reflected on the work of Fermat, and noted the relation of exponents and coefficients in simple differentiation, then stepped from this to partial differentiation. It may be significant that deSluse, like Fermat, used $a$ for the subtangent in [14], though he used $\eta$ and $\omega$, where we should use $x$ and $y$.

In connection with deSluse, it is necessary to say something about Barrow. He gives deSluse' method [14] in his [18], stating, modestly enough, that he gives his method at the request of a friend (presumably Newton), "Though I scarcely perceive the use of doing so, considering the several methods of this Nature now in use." His rule, as stated, means very little, but, worked out, it amounts to that of deSluse. In his first example, he commits what seems to me to be the most heinous possible mathematical sin. He uses the same letter to mean two different things in the same problem, writing the equation

$$qq - 2qe + mm - 2ma = BLq,$$

where the right side means $BL^2$.

Barrow's most interesting example is his fifth where $y=\tan x$. He takes an auxiliary circle of radius $r$, and finds the point with Cartesian coordinates

$(r \cos x, r \sin x)$. He then gives $x$ a small increment $dx$ and this he puts equal to $\frac{1}{2} \sin 2\ dx$. If then we compare an infinitesimal triangle with a finite one, we get the fundamental formula

$$d \cos x = - \sin x dx.$$

He finally reaches

$$\frac{dy}{dx} = 1 + y^2.$$

J. M. Child, whose admiration for Barrow seems to me a bit excessive, says in [19] "If $y = \tan x$, $dy/dx = \sec^2 x$." He must have known (for it is in itself evident) that the same two diagrams can be used for any of the trigonometric ratios. Therefore, Barrow must be credited with the differentiation of the circular functions. This is possible, but not at all certain. Eight years later James Bernoulli gave, as known, the formulae for the derivatives of tan $x$ and sec $x$. The credit for differentiating all six functions is usually assigned to Roger Cotes and his *Harmonia Mensurarum* of 1722. Child even assigns to Barrow the credit for inventing the process of differentiation.

I should mention, in connection with Barrow, two other theorems. The first is in his fourth lecture, and consists in a proof of Fermat's sub-tangent formula $a/y = dx/dy$. His proof, like the rest of his work, I find obscurely written. He works, not with infinitesimals, but with velocities. Next, let us take a convex arc and a point $M$ where the slope of the tangent is $l$. An ordinate shall move at a constant rate, cutting the curve at $O$ and the tangent at $K$. When $K$ and $O$ are above $G$, where the moving ordinate cuts the horizontal line through $M$, if $O$ is further from this horizontal line than is $K$, it must go further in the same time than does $K$. Its dropping velocity $dy/dt > l\ dx/dt$, where $dx/dt$ is the constant velocity of the ordinate. But, when the dropping velocity on the tangent is less than that on the curve, we have $dy/dt < l(dx/dt)$. Barrow concludes that at $M$, $dy/dt = l(dx/dt)$. I have naturally shortened the proof by using modern notation. Fermat's formula will come at once from this. I should mention that on page 61 of [19] we have the fundamental statement: *If the arc MN is assumed indefinitely small we may safely substitute instead of it a small bit of the tangent.* This is, of course, the basis of Fermat's process of "adequation" and, in fact, of most seventeenth century work with tangents, except that of the school of Roberval and Torricelli.

**5. Newton and Leibniz.** We have seen that Newton's method of drawing tangents was the same as that of deSluse. He probably discovered it in 1664 or 1665, when he was first thinking through the methods of the infinitesimal calculus. He published nothing before 1669, and then only in a letter. A complete discussion giving nine different examples appeared under Problem IV of [26]. He begins like deSluse. If we take two near points of a curve, and connect them by a straight line, which we treat as if it were the tangent, we have

two similar triangles. The one is bounded by the tangent, the sub-tangent and the ordinate; the other by the element of arc and what he calls the "moments" of the two coordinates, which are in the language of Leibniz the differences, and are proportional to their "fluxions." He would write Fermat's first formula

$$\frac{e}{y} = \frac{\dot{x}}{\dot{y}}.$$

The first example is

$$x^3 - ax^2y + axy - y^3 = 0,$$

$$3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + axy - 3\dot{y}y^2 = 0,$$

$$l = \frac{3y^3 - axy}{3x^2 - 2ax + ay}.$$

He gives other examples, sometimes introducing other variables, but nothing essentially different. Some of his methods show the influence of Roberval, or a like-thinking geometer. In the third memoir, he assumes that a point is known by $r_1$ and $r_2$, its distances from two fixed points. These are connected by a known equation, from which we find $\dot{r}_1/\dot{r}_2$. We draw a perpendicular to the second radius vector at its origin, and extend it till it meets the tangent, and thence drop a perpendicular on the first radius vector. If the distance from this last intersection to the point of contact be $S$, we have, by similar trapezoids, $s/r_2 = \dot{r}_1/\dot{r}_1$. Hence $s$ is known, and we can find a point on the tangent. In the same way, he handles tangents given in polar coordinates, finding the polar sub-tangent.

When it comes to Leibniz, I move with extreme caution, not wishing to express any opinion on the Newton-Leibniz priority question. Newton wrote in a letter to Oldenburg, October 24, 1676, intended for Leibniz, a long description of his methods, including a reference to deSluse's method of finding tangents. Leibniz answered [27] at once, maintaining that deSluse's methods are not in themselves sufficient, and that he himself had discovered a method. He adds, "Clarissimi Slusii Methodum Tangentium nodum esse absolutam celeberrimo Newtono assentior. Et jam a multo tempore rem Tangentium longe generalius tractavi, scilicet per differentia Ordinaturum."

The essential part is found here in the words, "Longe generalius tractavi." He says that for a long time he had treated tangents by the differences of ordinates, the differences of abscissas were treated as constants. He gives various examples. If we disregard higher infinitesimals, we have

$$dy^2 = 2ydy, \qquad dy^2x = 2xydy + y^2dx,$$

given

$$a + by + cx + dyx + ey^2 + fx^2 + \cdots = 0,$$

$$-\frac{dy}{dx}=\frac{c+dy+2fx+\cdots}{b+dx+2ey+\cdots}.$$

"Quod coincidit cum Regular Slussiana." He writes as a universally accepted principle; "Si sit aliqua potentia aut radix $x^z$ erit $dx^z=zx^{z-1}dx$.

To find $d\sqrt[3]{a+by+cy^2+\cdots}$ he puts

$$z=1/3; \qquad x=a+by+cy^2+\cdots$$

$$dx^z=\frac{dx}{3x^{2/3}}=\frac{bdy+2cydy+\cdots}{3(a+by+cy^2+\cdots)^{2/3}}.$$

Newton had written to Collins that he did not wish to show his method of tangents to Leibniz. The letter ended with the following, "Arbitrior quae celare voluit Newtonus de Tangentibus ducendis, ab his non abludere." I see no point in going further into the famous Newton-Leibniz priority controversy.

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## A NEW ABSOLUTE GEOMETRIC CONSTANT?

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In Whitworth, *Choice and Chance*, appears the following problem: If there be $n$ straight lines in one plane, no three of which meet in a point, the number of groups of $n$ of their points of intersection, in each of which no three points lie in one of the straight lines, is $\frac{1}{2}(n-1)!$

We shall assume either that no two lines are parallel, or that the problem refers to the extended plane. To prove that there are just two points on each line, note that since each of the $n$ points lies on two lines, there are $2n$ incidences in the set, which is an average of 2 incidences per line. But since no line may contain as many as 3 incidences, each must contain just 2.

The problem may now be regarded as requiring the number of polygons of $n$ sides formed by a plane network of $n$ lines, no three concurrent. Whitworth's answer is correct only if no polygon is allowed to consist of two or more polygons of fewer sides, *e.g.*, a composite polygon consisting of two polygons of $k$ and $n-k$ sides, respectively. There is, however, nothing in his statement of the problem to rule out such cases.

Let $g_n$ be the number of $n$-gons formed by a network of $n$ lines. We shall show that $g_n$ becomes infinite like $\sqrt{n}(n-1)!$, which will correct Whitworth's answer. To be more precise, we shall show that $g_n$ becomes infinite like $n^n e^{-n}$, in other words, that

$$\lim_{n\to\infty} \frac{g_n}{n^n e^{-n}} = B,$$

where $B$ is a constant.

Let us proceed by induction. Suppose we raise the number of lines from $(n-1)$ to $n$ by adding one line to $(n-1)$ already given. Each $(n-1)$-gon can be converted into an $n$-gon by the following operation: Replace any one vertex by the two points where the two sides through it meet the new line. Conversely,

an $n$-gon which can be so obtained can be evolved by this operation from only one $(n-1)$-gon, and by replacement of only one vertex, since the operation is reversible uniquely. For example, in Figure 1, the 4-gon $ABCD$ may be converted into the 5-gon $ABCD_1D_2$, when the fifth line $D_1D_2$ is added; this is the only way that the 5-gon $ABCD_1D_2$ may be obtained in this manner from a 4-gon formed by the four lines originally given.

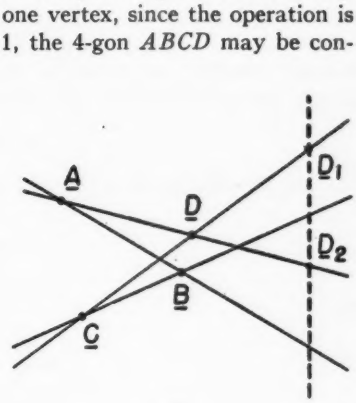


FIG. 1

If these $n$-gons were the only ones in the new network of $n$ lines, then we should have $g_n=(n-1)g_{n-1}$, since one $n$-gon corresponds to each vertex of each $(n-1)$-gon. Since obviously $g_3=1$, we should have

$$g_n=(n-1)g_{n-1}=(n-1)(n-2)g_{n-2}=\cdots$$
$$=(n-1)(n-2)\cdots 3g_3=\tfrac{1}{2}(n-1)!,$$

which is Whitworth's answer. Unfortunately for Whitworth, there are $n$-gons which cannot be obtained by this operation, namely, every composite $n$-gon one of whose parts is a triangle having the $n$th line as one side. For example, in Figure 2, the 6-gon $ABCDEF$ consists of the triangles $ABC$ and $DEF$. If the line $EF$ is removed, allowing $E$ and $F$ to be replaced by $D$ in the reverse of the operation defined above, the result consists of the triangle $ABC$ and the vertex $D$ counted twice, which is not a 5-gon at all.

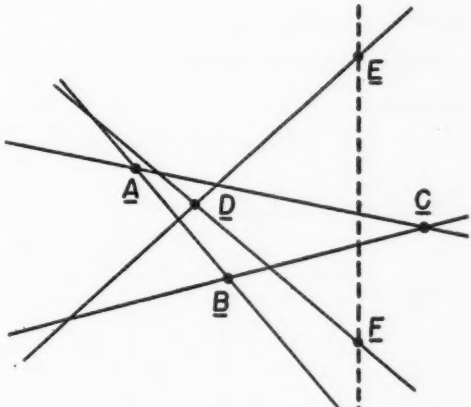


FIG. 2

These $n$-gons are, however, the only exceptions, and can be counted. On the $n$th line there are just $(n-1)$ possible vertices, a pair of which can be chosen in $C_2^{n-1}$ ways. When a pair is chosen, the triangle is determined, and the remainder of the composite $n$-gon may be chosen from the $(n-3)$-gons on the remaining $(n-3)$ lines in $g_{n-3}$ ways. Hence,

$$g_n=(n-1)g_{n-1}+C_2^{n-1}g_{n-3},$$

or, more conveniently,

$$g_{n+1} = ng_n + C_2^n g_{n-2}. \tag{1}$$

We may simplify our work by introducing $u_n$, the ratio of $g_n$ to Whitworth's answer for non-composite polygons:

$$g_n = u_n \cdot \tfrac{1}{2}(n-1)!$$

Making this change in (1), and dividing by $\frac{1}{2}n!$,

$$u_{n+1} = u_n + \frac{1}{2(n-2)} u_{n-2},$$

or,

$$u_{n+3} = u_{n+2} + \frac{1}{2n} u_n. \tag{2}$$

Since no composite polygons exist with less than 6 sides, $u_3 = u_4 = u_5 = 1$. Necessarily $n \geqq 3$.

Although this recursion formula is not readily solved for $u_n$ explicitly in terms of $n$, we can nevertheless determine quite a bit about the behavior of $u_n$ as $n$ becomes large.

LEMMA 1. $u_{n+1} > u_n$, $n > 4$.

This is obvious from (2).

THEOREM 1.

*The number* $b = \lim_{n\to\infty} \dfrac{u_n^2}{n - \frac{9}{4}}$ *exists, and* $b \geqq 0$.

Since $u_n < u_{n+1} < u_{n+2}$,

$$u_{n+3} = u_{n+2} + \frac{1}{2n} u_n < u_{n+2} + \frac{1}{2n} u_{n+2},$$

$$u_{n+3}^2 < u_{n+2}^2 \left(1 + \frac{1}{2n}\right)^2 = u_{n+2}^2 \left[1 + \frac{1}{n}\left(1 + \frac{1}{4n}\right)\right],$$

$$u_{n+3}^2 < u_{n+2}^2 \left[1 + \frac{1}{n}\left(1 + \frac{1}{4n} + \frac{1}{(4n)^2} + \cdots\right)\right],$$

$$u_{n+3}^2 < u_{n+2}^2 \left[1 + \frac{1}{n} \cdot \frac{1}{1 - \frac{1}{4n}}\right] = u_{n+2}^2 \frac{n + \frac{3}{4}}{n - \frac{1}{4}},$$

$$0 < \frac{u_{n+3}^2}{n + \frac{3}{4}} < \frac{u_{n+2}^2}{n - \frac{1}{4}} < \frac{u_n^2}{n - \frac{9}{4}}.$$

Since $u_n^2/(n-\frac{9}{4})$ decreases as $n$ increases, and is always positive, it has a limit, positive or zero.

THEOREM 2.

$$\lim_{n\to\infty} \frac{u_n^2}{n-i} = b,$$

*where $i$ is any constant.*

This follows from

$$\lim_{n\to\infty} \frac{n-\frac{9}{4}}{n-i} = 1,$$

and Theorem 1.

LEMMA 2.

$$u_n > u_{n+2}\left(1 - \frac{1}{n-\frac{9}{4}}\right).$$

In proving Theorem 1, we showed that

$$\frac{u_{n+2}^2}{n-\frac{1}{4}} < \frac{u_n^2}{n-\frac{9}{4}}.$$

$$u_n^2 > u_{n+2}^2 \frac{n-\frac{9}{4}}{n-\frac{1}{4}} = u_{n+2}^2\left(1 - \frac{2}{n-\frac{1}{4}}\right).$$

Since $n \geqq 3$, $2/(n-\frac{1}{4}) < 1$, we may write

$$u_n > u_{n+2}\left(1 - \frac{2}{n-\frac{1}{4}}\right)^{1/2}$$

$$u_n > u_{n+2}\left[1 - \tfrac{1}{2}\left(\frac{2}{n-\frac{1}{4}}\right) - \tfrac{1}{8}\left(\frac{2}{n-\frac{1}{4}}\right)^2 - \tfrac{1}{16}\left(\frac{2}{n-\frac{1}{4}}\right)^3 - \cdots\right].$$

Note that all coefficients after the second term of the series are numerically less than $\frac{1}{2}$, and negative:

$$\pm \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots}{1\cdot 2\cdot 3\cdot 4\cdots}.$$

So

$$u_n > u_{n+2}\left\{1 - \frac{1}{n-\frac{1}{4}}\left[1 + \tfrac{1}{4}\left(\frac{2}{n-\frac{1}{4}}\right) + \tfrac{1}{8}\left(\frac{2}{n-\frac{1}{4}}\right)^2 + \cdots\right]\right\}.$$

Now all the coefficients in the brackets are positive and $\leqq 1$, so

$$u_n > u_{n+2}\left\{1 - \frac{1}{n-\frac{1}{4}}\left[1 + \left(\frac{2}{n-\frac{1}{4}}\right) + \left(\frac{2}{n-\frac{1}{4}}\right)^2 + \cdots\right]\right\},$$

$$u_n > u_{n+2}\left(1 - \frac{1}{n-\frac{1}{4}} \cdot \frac{1}{1-\frac{2}{n-\frac{1}{4}}}\right) = u_{n+2}\left(1 - \frac{1}{n-\frac{9}{4}}\right).$$

LEMMA 3. *If* $n \geqq 13$, then

$$\frac{1}{n} - \frac{\frac{5}{4}}{n(n-\frac{9}{4})} > \frac{1}{n+\frac{7}{4}}.$$

$$n \geqq 13 > \tfrac{49}{4},$$

$$7n - 5n = 2n > \tfrac{98}{4} = \tfrac{63}{4} + \tfrac{35}{4}.$$

$$7n - \tfrac{63}{4} > 5n + \tfrac{35}{4},$$

$$7(n-\tfrac{9}{4}) > 5(n+\tfrac{7}{4}),$$

$$\frac{7}{n+\frac{7}{4}} > \frac{5}{n-\frac{9}{4}},$$

$$\frac{1}{n} - \frac{1}{n+\frac{7}{4}} = \frac{\frac{7}{4}}{n(n+\frac{7}{4})} > \frac{\frac{5}{4}}{n(n-\frac{9}{4})},$$

$$\frac{1}{n} - \frac{\frac{5}{4}}{n(n-\frac{9}{4})} > \frac{1}{n+\frac{7}{4}}.$$

THEOREM 3. $b>0$.

By Lemma 2,

$$u_{n+3} = u_{n+2} + \frac{1}{2n}u_n > u_{n+2} + \frac{1}{2n}u_{n+2}\left(1 - \frac{1}{n-\frac{9}{4}}\right),$$

$$u_{n+3} > u_{n+2}\left[\left(1 + \frac{1}{2n}\right) - \frac{1}{2n(n-\frac{9}{4})}\right],$$

$$u_{n+3}^2 > u_{n+2}^2\left[\left(1 + \frac{1}{n} + \frac{1}{4n^2}\right) - \frac{1+\frac{1}{2n}}{n(n-\frac{9}{4})} + \frac{1}{4n^2(n-\frac{9}{4})^2}\right].$$

Since $n>2$, $1/2n<\frac{1}{4}$, and

$$u_{n+3}^2 > u_{n+2}^2\left[1 + \frac{1}{n} - \frac{\frac{5}{4}}{n(n-\frac{9}{4})}\right].$$

By Lemma 3,

$$u_{n+3}^2 > u_{n+2}^2\left[1 + \frac{1}{n+\frac{7}{4}}\right] = u_{n+2}^2 \frac{n+\frac{11}{4}}{n+\frac{7}{4}}, \qquad n \geqq 13.$$

$$\cdots > \frac{u_{n+3}^2}{n+\frac{11}{4}} > \frac{u_{n+2}^2}{n+\frac{7}{4}} > \frac{u_n^2}{n-\frac{1}{4}} > 0, \qquad n \geqq 15.$$

Hence

$$b = \lim_{n\to\infty} \frac{u_n^2}{n-\frac{1}{4}} > 0.$$

THEOREM 4.

$$\lim_{n\to\infty} (u_{n+1}^2 - u_n^2) = b.$$

$$u_{n+3}^2 - u_{n+2}^2 = (u_{n+3} - u_{n+2})(u_{n+3} + u_{n+2})$$

$$= \frac{u_n}{2n}(u_{n+3} + u_{n+2})$$

$$= \frac{1}{2}\frac{u_n}{\sqrt{n}}\left(\frac{u_{n+3}}{\sqrt{n}} + \frac{u_{n+2}}{\sqrt{n}}\right).$$

$$\lim_{n\to\infty} (u_{n+3}^2 - u_{n+2}^2) = \tfrac{1}{2}\sqrt{b}(\sqrt{b} + \sqrt{b}) = b.$$

Incidentally, we have shown that $u_n^2/(n-\frac{9}{4})$ always decreases, while $u_n^2/(n-\frac{1}{4})$ always increases beyond a certain point. The author has strengthened these theorems considerably in work not included here, and from these results and a study of numerical tables, it seems virtually certain that $u_n^2/(n-i)$ always decreases, $(n>6)$ when $i \geqq \frac{5}{4}$, but always increases beyond a certain point $(n>N_i)$ when $i<\frac{5}{4}$. However, apparently, $\lim_{i\to\frac{1}{4}} N_i = \infty$. If these facts are true, it can be shown that

$$\frac{u_n^2}{n-\frac{5}{4}} > u_{n+1}^2 - u_n^2 > \frac{u_n^2}{n-i}, \qquad i < \tfrac{5}{4},$$

the first inequality being true for $n>6$, the second for $n>N_i$.

These conjectures are valuable chiefly because they aid in the numerical computation of the limit $b$. Suppose we tabulate values of $u_{n+1}^2 - u_n^2$ and $u_n^2/(n-\frac{5}{4})$. The following is an excerpt from such a table:

| $n$ | $u_{n+1}^2 - u_n^2$ | $u_n^2/(n-\frac{5}{4})$ |
|---|---|---|
| 21 | 0.283,797 | 0.284,405 |
| 22 | 0.283,825 | 0.284,376 |
| 23 | 0.283,850 | 0.284,350 |
| 24 | 0.283,871 | 0.284,328 |
| 25 | 0.283,890 | 0.284,309 |
| 26 | 0.283,907 | 0.284,292 |
| 27 | 0.283,921 | 0.284,277 |
| 28 | 0.283,934 | 0.284,264 |
| 29 | 0.283,946 | 0.284,252 |
| 30 | 0.283,956 | 0.284,241 |

The tabular differences of these values are:

| $\Delta_1$ | $\Delta_2$ |
|---|---|
| 33 | 34 |
| 28 | 29 |
| 25 | 26 |
| 21 | 22 |
| 19 | 19 |
| 17 | 17 |
| 14 | 15 |
| 13 | 13 |
| 12 | 12 |
| 10 | 11 |

Since one of the quantities is decreasing toward $b$, the other increasing, the similarity of the tabular differences suggests that the average of the values in the two columns constitutes a good first approximation to $b$. This value for $n=30$ is 0.284,098.

We are now in a position to write as an asymptotic expression,

$$u_n \cong \sqrt{b(n-\tfrac{5}{4})}.$$

Using this with Stirling's formula, that is,

$$n! \cong n^n e^{-n}\sqrt{2\pi(n+\tfrac{1}{6})},$$

we may also write

$$g_{n+1} = u_{n+1}\cdot\tfrac{1}{2}n! \cong n^n e^{-n}\sqrt{\tfrac{1}{2}\pi b(n+\tfrac{1}{6})(n-\tfrac{1}{4})}.$$

Only slightly less accurately, but more simply,

$$g_{n+1} \cong n^{n+1}e^{-n}\sqrt{\tfrac{1}{2}\pi b}.$$

From this,

$$g_n \cong (n-1)^n e^{-(n-1)}\sqrt{\tfrac{1}{2}\pi b} = n^n e^{-n}\sqrt{\tfrac{1}{2}\pi b}\cdot e\left(1-\frac{1}{n}\right)^n.$$

Since

$$\lim_{n\to\infty} e\left(1-\frac{1}{n}\right)^n = 1,$$

we have, setting $B=\sqrt{\tfrac{1}{2}\pi b}$,

$$g_n \cong B\cdot n^n e^{-n},$$

which was the contention at the start of the paper. The numerical value of $B$ is 0.668,027, but this is no more illuminating than the numerical value of $b$.

By improving the asymptotic expression by series expansion corrections,

the author has been able to improve the numerical value for $b$ by several decimal places; but this greater accuracy seems to be of slight use, unless perchance it may suggest further avenues of approach to the true nature of $b$.

Thus $b$ assumes the role of an absolute geometric constant. The author is interested in discovering whether it is an algebraic number, which seems unlikely, whether it is expressible in terms of $\pi$ and $e$, or whether it is entirely new. Perhaps some reader of this discussion will be able to throw light on the subject by use of other methods.

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## HYPERBOLIC TRIGONOMETRY DERIVED FROM THE POINCARÉ MODEL*

HOWARD EVES and V. E. HOGGATT, JR., Oregon State College

**1. Introduction.** The formulas of hyperbolic trigonometry have been derived in a number of ingenious ways. As early as 1766 Lambert [1] suggested that the geometry of the "third hypothesis" could be verified on a sphere of imaginary radius. The historical process, developed by Bolyai and Lobachewsky, made use of the fact that the geometry of horocycles on the horosphere is euclidean in nature [2]. Sommerville has given an excellent elementary treatment along these lines [3]. Some early writers, however, regretted any appeal to solid geometry in the derivation of formulas for a plane trigonometry. Clever methods were devised to remedy the defect, one of the neatest being due to Liebmann [4], and subsequently reproduced by such textbook writers as Carslaw [5] and Wolfe [6]. Other ingenious methods were furnished by Gérard [7], Young [8], and, more recently, by Fulton [9]. Coolidge [10] supplied a careful treatment based upon the fact that hyperbolic geometry is euclidean in the small.

In order to establish the relative consistency of the hyperbolic and euclidean geometries, it suffices to devise a model in euclidean space containing elements, with appropriate connecting relations, which, when substituted for the undefined elements and relations of a postulate set for hyperbolic geometry, will interpret those postulates as true theorems of euclidean geometry. Many such models have been devised, the most famous ones being due to Beltrami, Cayley, Klein, and Poincaré [11]. Once such a model has been formed it is conceivable that some theorems of hyperbolic geometry might be more readily established by demonstrating the euclidean analogues in the model rather than the originals directly from the accepted postulate set. It is the purpose of this paper to de-

* A fuller development of the ideas of this paper may be found in V. E. Hoggatt's Oregon State College Masters Thesis, 1951.

velop the formulas of hyperbolic plane trigonometry in such a manner, employing the elementary model usually referred to as the *Poincaré model* [12]. Carslaw has shown the utility of this particular model by very simply establishing from it several difficult theorems of hyperbolic geometry [13].

**2. Description of the Poincaré model.** As an acceptable postulate set for plane hyperbolic geometry one could select that of Hilbert [14]. The primitive terms of this postulate set are *point, line, on* (applied to a point and a line), *between* (applied to three points on a line), and *congruent* (applied to segments and angles). The postulates are statements concerning these primitive terms. A euclidean model of plane hyperbolic geometry must, then, be a system of geometrical elements and relations which, when substituted for the primitive terms, convert the Hilbert postulates into true theorems in euclidean geometry. The Poincaré model accomplishes this as follows.

A fixed circle $\Sigma$ is selected and called the *fundamental circle*. A point of the hyperbolic plane is represented by a point interior to $\Sigma$, hereafter called a *nominal point*. A line of the hyperbolic plane is represented by the arc interior to $\Sigma$ of any circle orthogonal to $\Sigma$, hereafter called a *nominal line*. The relationships of a point on a line, and a point between two points, have the obvious interpretations. To suitably interpret congruence of segments and congruence of angles two definitions are made. The (positive) *nominal length* of a nominal segment $AB$ is defined as

$$\overline{AB} = \log_a (AB, TS) = k \ln (AB, TS), \qquad k = -\ln a,$$

where $S$ and $T$ are the points where the nominal line $AB$ meets $\Sigma$, $A$ lying between $S$ and $B$, and $(AB, TS)$ denotes the cross-ratio $(AT/BT)/(AS/BS)$ of the circular range $A, B, T, S$. And the *nominal measure* of the angle between two intersecting nominal lines is defined as the ordinary radian measure of the angle between the two circles on which the nominal lines lie. Two nominal segments are congruent if and only if they have equal nominal lengths, and two nominal angles are congruent if and only if they have equal nominal measures.

It can be shown that with the above interpretations the Hilbert postulates for plane hyperbolic geometry become true theorems in euclidean geometry. For every theorem in hyperbolic geometry there is the euclidean counterpart in the Poincaré model, and the establishment of the latter carries with it that of the former. We now proceed to establish hyperbolic trigonometry by obtaining the necessary counterparts in the Poincaré model.

**3. Preliminary theorems.** Consider any nominal right triangle $O'P'Q'$, right angled at $Q'$ (see Fig. 1), and let the circles determined by the nominal lines $O'P'$ and $O'Q'$ intersect again in $C$. Invert the figure with respect to $C$ as center and with a power that carries $\Sigma$ into itself. Since inversion is a conformal transformation, the circles $CP'O'$, $CQ'O'$, being orthogonal to $\Sigma$ and passing through the center of inversion $C$, invert into two diametral lines of $\Sigma$. Thus, by the

inversion, the right triangle $O'P'Q'$ is carried into the right triangle $OPQ$, where $OP$ and $OQ$ are radial lines of $\Sigma$. Since both angles and cross-ratios are preserved under inversion, it follows that nominal triangles $O'P'Q'$ and $OPQ$ are nominally congruent, and, to obtain the fundamental formulas of hyperbolic plane trigonometry, it suffices to study the relations connecting the nominal lengths of the sides and the nominal measures of the angles of the specially placed right triangle $OPQ$. We shall consistently distinguish euclidean lengths from nominal lengths by placing bars over the latter. Since angles have the same nominal and euclidean measures, no bars are here needed.

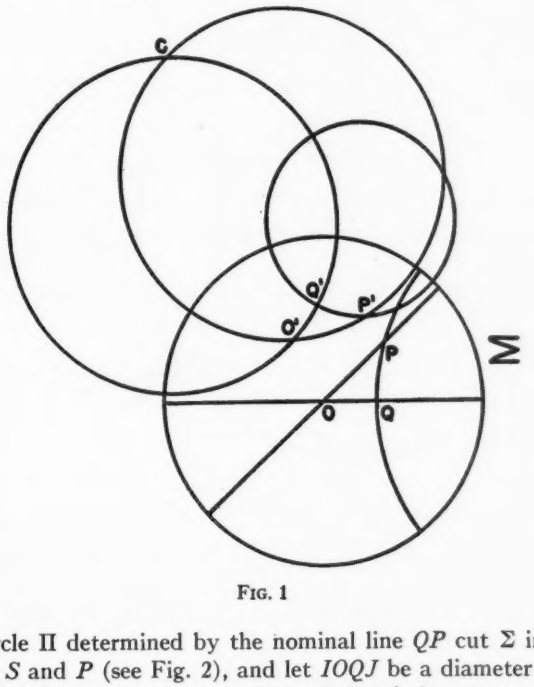


FIG. 1

Let the circle $\Pi$ determined by the nominal line $QP$ cut $\Sigma$ in $S$ and $T$, $Q$ lying between $S$ and $P$ (see Fig. 2), and let $IOQJ$ be a diameter of $\Sigma$, cutting $\Pi$ again in $W$. We now establish a short chain of theorems connected with Figure 2.

THEOREM 3.1. *If $WS$ and $WT$ cut $\Sigma$ again in $U$ and $V$, then $UV$ is the diameter of $\Sigma$ perpendicular to diameter $IJ$.*

Select $W$ as center of inversion, and choose a power such that $\Sigma$ inverts into itself. Then $S$ inverts into $U$, and $T$ into $V$. Since $\Pi$ is orthogonal to both $\Sigma$ and $IJ$, it follows that $UV$ is the diameter of $\Sigma$ perpendicular to diameter $IJ$.

THEOREM 3.2. *Let $WP$ cut $UV$ in $R$, and designate the lengths of $OW$ and $OR$*

*by $m$ and $n$, and the radius of $\Sigma$ by $r$. Let $K$ be the center of $\Pi$ and let $M$ and $N$ be the feet of the perpendiculars dropped from $P$ on $OW$ and $OR$ respectively. Then*

(a) $KP = (m^2 - r^2)/2m$, (b) $OM = m(n^2 + r^2)/(m^2 + n^2)$,

(c) $OP = (m^2n^2 + r^4)^{1/2}/(m^2 + n^2)^{1/2}$, (d) $OQ = r^2/m$.

Since $KP = OW - OK = m - (r^2 + KP^2)^{1/2}$, it follows that

$$KP = (m^2 - r^2)/2m.$$

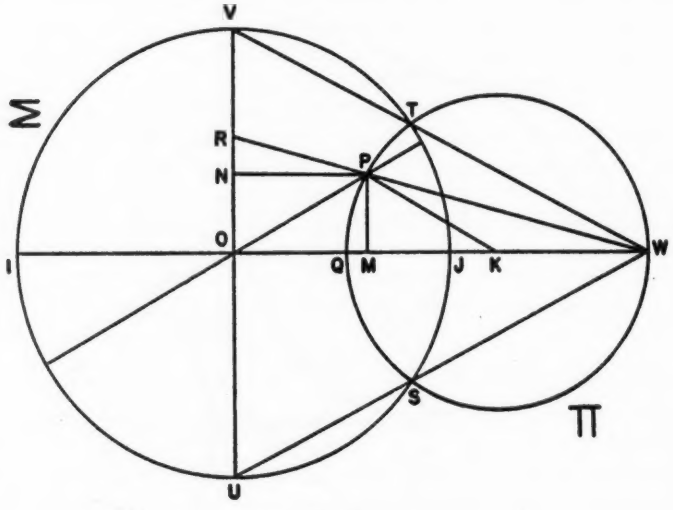


FIG. 2

Also, since $\tan PWO = n/m$, and since $\sphericalangle PKO = 2\sphericalangle PWO$, it follows that

$$\tan PKO = 2mn/(m^2 - n^2), \qquad \sin PKO = 2mn/(m^2 + n^2).$$

Therefore

$$MP = KP \sin PKO = n(m^2 - r^2)/(m^2 + n^2).$$

And, from similar triangles $RNP$ and $ROW$,

$$OM = NP = (OW)(NR)/OR = m(n - MP)/n = m(n^2 + r^2)/(m^2 + n^2).$$

Then

$$OP^2 = MP^2 + OM^2 = (m^2n^2 + r^4)/(m^2 + n^2).$$

Finally,

$$OQ = OW - 2KP = m - 2(m^2 - r^2)/2m = r^2/m.$$

THEOREM 3.3. *The segments OP, OQ, OR are connected by the relation*

$$\frac{r^2+OP^2}{r^2-OP^2}=\frac{r^2+OR^2}{r^2-OR^2}\cdot\frac{r^2+OQ^2}{r^2-OQ^2}.$$

For, by theorem 3.2 (c),

$$\frac{r^2+OP^2}{r^2-OP^2}=\frac{r^2(m^2+n^2)+m^2n^2+r^4}{r^2(m^2+n^2)-m^2n^2-r^4}=\frac{(r^2+n^2)(m^2+r^2)}{(r^2-n^2)(m^2-r^2)}$$
$$=\frac{r^2+n^2}{r^2-n^2}\cdot\frac{r^2+r^4/m^2}{r^2-r^4/m^2}=\frac{r^2+OR^2}{r^2-OR^2}\cdot\frac{r^2+OQ^2}{r^2-OQ^2},$$

since $OR=n$ and, by theorem 3.2 (d), $OQ=r^2/m$.

THEOREM 3.4. *If OQ is any radial segment of* $\Sigma$, *then*

(a) $$\cosh(\overline{OQ}/k)=(r^2+OQ^2)/(r^2-OQ^2),$$

(b) $$\sinh(\overline{OQ}/k)=2rOQ/(r^2-OQ^2),$$

(c) $$\tanh(\overline{OQ}/k)=2rOQ/(r^2+OQ^2).$$

For, since $\overline{OQ}/k=ln(OQ, IJ)$, we have

$$\begin{aligned}\cosh(\overline{OQ}/k)&=[\exp(\overline{OQ}/k)+\exp(-\overline{OQ}/k)]/2\\&=[(OQ, IJ)+(OQ, JI)]/2\\&=[(OI/QI)/(OJ/QJ)+(OJ/QJ)/(OI/QI)]/2\\&=[QJ/IQ+IQ/QJ]/2\\&=[(r-OQ)/(r+OQ)+(r+OQ)/(r-OQ)]/2\\&=(r^2+OQ^2)/(r^2-OQ^2),\end{aligned}$$

and relation (a) is established. Relations (b) and (c) follow in a similar manner, or from the identities $\sinh^2 x=\cosh^2 x-1$ and $\tanh x=(\sinh x)/(\cosh x)$.

**4. Hyperbolic plane trigonometry.** We are now ready to derive the formulas of hyperbolic plane trigonometry. It is well known that the formulas for the general hyperbolic triangle are readily derived from those for the hyperbolic right triangle. Let us be given such a right triangle $ABC$, right angled at $C$, and designate the lengths of the sides opposite $A$, $B$, $C$ by $a$, $b$, $c$, and let $k$ be the parameter of hyperbolic geometry. Then it is easily shown that all the remaining formulas for the right triangle can be obtained by purely analytical procedures from the following two:

$$\cosh c/k=\cosh a/k\cosh b/k,$$
$$\cos A=(\tanh b/k)/(\tanh c/k).$$

It therefore remains for us to establish these two formulas, and we accomplish this by proving the following two theorems.

THEOREM 4.1. *In figure* 2, $\cosh(\overline{OP}/k) = \cosh(\overline{OQ}/k)\cosh(\overline{QP}/k)$.

As an immediate consequence of Theorems 3.3 and 3.4 (a) we have

$$\cosh(\overline{OP}/k) = \cosh(\overline{OQ}/k)\cosh(\overline{OR}/k).$$

But, by Theorem 3.1, $(QP, ST) = W(QP, ST) = (OR, UV)$, whence $\overline{OR} = \overline{QP}$, and the theorem is established.

THEOREM 4.2. *In Figure* 2, $\cos QOP = \tanh(\overline{OQ}/k)/\tanh(\overline{OP}/k)$.

For, by theorem 3.4 (c),

$$\tanh(\overline{OQ}/k)/\tanh(\overline{OP}/k) = OQ(r^2 + OP^2)/OP(r^2 + OQ^2).$$

Substituting the expressions for $OP$ and $OQ$ as given by Theorem 3.2 (c) and (d), and simplifying, we find

$$\begin{aligned}\tanh(\overline{OQ}/k)/\tanh(\overline{OP}/k) &= m(r^2 + n^2)/(m^2n^2 + r^4)^{1/2}(m^2 + n^2)^{1/2} \\ &= [m(r^2 + n^2)/(m^2 + n^2)]/[(m^2n^2 + r^4)^{1/2}/(m^2 + n^2)^{1/2}] \\ &= OM/OP \qquad \text{(Theorems 3.2 (b) and (c))} \\ &= \cos QOP,\end{aligned}$$

and the theorem is established.

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## THE BETA FUNCTION

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The First Eulerian Integral, called the Beta Function, is defined by $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$, which converges for $x>0$ and $y>0$. The well-known equation connecting the Beta and Gamma functions, $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$, is therefore valid only for positive real $x$ and $y$. However, this relation is commonly used as a definition, to extend $B(x, y)$ so that the function has meaning for values other than positive real $x$, $y$. It is the purpose of this paper to discuss the behavior of $B(x, y)$ over all real $x$ and $y$ for which the function is thus defined.

The discussion is greatly facilitated by graphic aids. Since $B(x, y)$ is continuous (with isolated exceptions) over the regions of definition, it can be represented by a Cartesian surface, $z = B(x, y)$. In particular, $z$ is continuous for all $x>0$, $y>0$, and so is represented by a smooth unbroken surface throughout the first octant (Fig. 1). Since it is obvious from the extended definition that $B(x, y)$

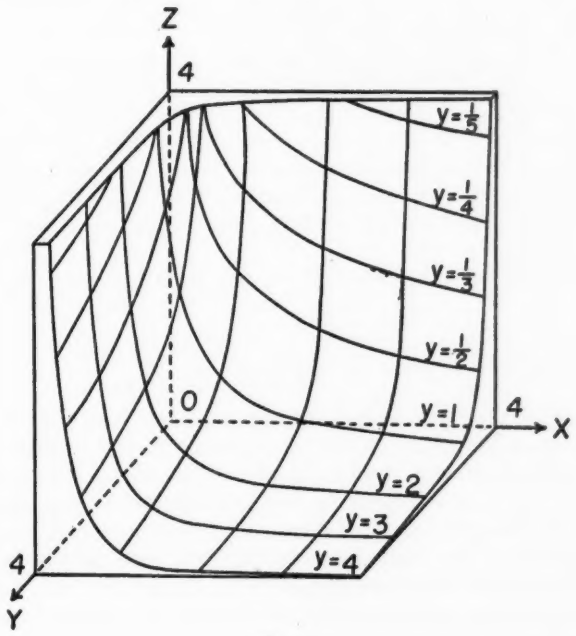


FIG. 1

$= B(y, x)$, we have symmetry with respect to the plane $y=x$, a property which also holds elsewhere than in the first octant. The surface passes through the point (1, 1, 1) but its asymptotic approach to the plane $z=0$ is much more

rapid than its approach to the planes $x=0$ and $y=0$. The value of $B(4, 4)$ is only $1/140$. It may be noted in passing that $B(\frac{1}{2}, \frac{1}{2})=\pi$.

When either or both of $(x, y)$ are allowed to assume negative values, the tracing of the surface becomes a much more lively affair. The surface is no longer everywhere continuous. Indeed there are points where it is not even defined. $\Gamma(x)$ oscillates between finite positive and negative values for all $x<0$, except that $\Gamma(-n)=\infty$ for all $n=1, 2, 3, \cdots$. Hence a vigorous oscillation of Beta, a combination of two Gammas, is to be expected in the negative range.

Let us first determine the points, if any, where $z=0$. $B(x, y)=0$ whenever $\Gamma(x+y)=\infty$ but $\Gamma(x)$ and $\Gamma(y)$ are both finite. Obviously this requires that the

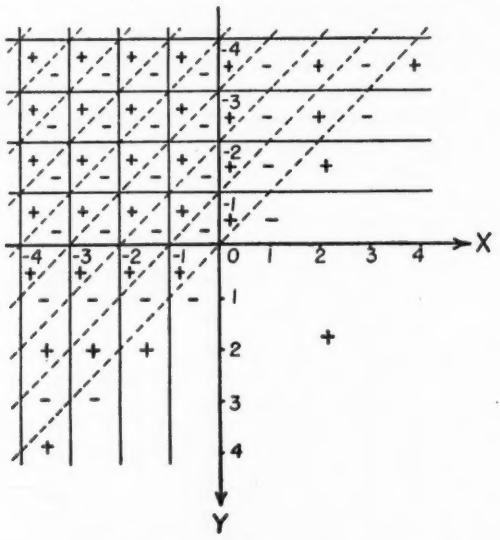


FIG. 2

sum $x+y$ be a negative integer but that neither $x$ nor $y$ be a negative integer. Samples are $B(-1/3, -2/3)=0$ and $B(3/2, -7/2)=0$. To indicate the locus of all such points we need another diagram. Figure 2 represents the plane $z=0$. $Y$ is taken positive downward to conform with the orientation of Figure 1. The plus and minus signs indicate whether the surface $z$ is above or below the plane of the paper in the region in question. Thus the whole first quadrant is marked $+$. The dotted diagonals represent the entirety of points where $z=0$. That is, the surface passes through the plane of the paper along every diagonal.

If $\Gamma(x)$ or $\Gamma(y)$ is infinite but $\Gamma(x+y)$ is finite, then $z=\infty$. This will occur when either $x$ or $y$ is a negative integer, but not both. Therefore along every horizontal and vertical line of Figure 2 the surface $z=B(x, y)\rightarrow\infty$ asymptotically. Whether it approaches infinity through positive or negative values of $z$ is governed by the sign of the region. Thus the planes $x=0, -1, -2, \cdots,$

and $y=0, -1, -2, \cdots$, divide the entire coordinate 3-space into cells, whose walls the surface cannot cross continuously. In the octants where $x$ and $y$ are both negative, each cell is a unit square column, closed on four sides and extending to infinity in the $z$ direction only; in the other octants the cells have one side open as well.

If both $x$ and $y$ are negative integers, or if both are integers and one is negative with absolute value $\geqq$ the other, then $\Gamma(x)\Gamma(y) = \infty$ and also $\Gamma(x+y) = \infty$. These are the only points where $B(x, y)$ is not defined, and they are the points in Figure 2 where any diagonal line crosses a horizontal or vertical line, inasmuch as the function cannot equal zero and infinity simultaneously. Thus the function is defined everywhere except at each corner of every cell.

At the risk of undue repetition, let us note again that $z$ is never discontinuous within any one cell. In order to preserve continuity in the long open cells of the lower left octants of Figure 2, the surface undergoes various twists and contortions. These cells are reproduced in mirror image across the plane $y=x$, and we shall therefore describe both sets by describing one. Each diamond-shaped region (of the lower left set), if extended in the $z$ direction perpendicular to the plane of the paper, contains a saddle, which approaches infinity along its right and left edges (+ or − as indicated), but merges smoothly into the next saddle along the dotted lines.

At each point of non-definition, the surface contains the line through that point perpendicular to the plane $z=0$. Thus the geometric interpretation of the lack of definition is that $z$ takes on all values from $+\infty$ to $-\infty$ at once.

The behavior of $B(x, y)$ in the octants where both $x$ and $y$ are negative is relatively simple. In half of every cell $z$ is plus, in the other half minus, as marked.

Finally an examination of a few plane sections $y=k$ may be instructive. The relation

$$B(x, y) = \frac{(y-1)!}{x(x+1)(x+2)\cdots(x+y-1)}$$

holds whenever $y$ is a positive integer, $x$ unrestricted. This emphasizes that the traces in Figure 1 found by setting $y=k$ are each part of some curve which has another branch or branches to the left of $x=0$. This last observation applies whether $k$ is integral or not.

If $y=1$ we have $B(x, 1) = z = 1/x$, the equilateral hyperbola of Figure 3($a$), the upper branch of which appears as a trace in Figure 1. The reader may wonder how the lower branch, a continuous curve, can successfully pierce all the barriers of discontinuity to the left of $x=0$. It accomplishes this feat by the simple expedient of meeting each barrier exactly at a cell corner, where the function is not defined. Thus the lower branch of Figure 3($a$) does not everywhere represent the function $B(x, 1)$. At $x=-1, -2, \cdots$, as well as at $x=0$, it fails to do so.

If $y=2$, $z=1/x(x+1)$, (Fig. 3($b$)). This trace is symmetric with respect to

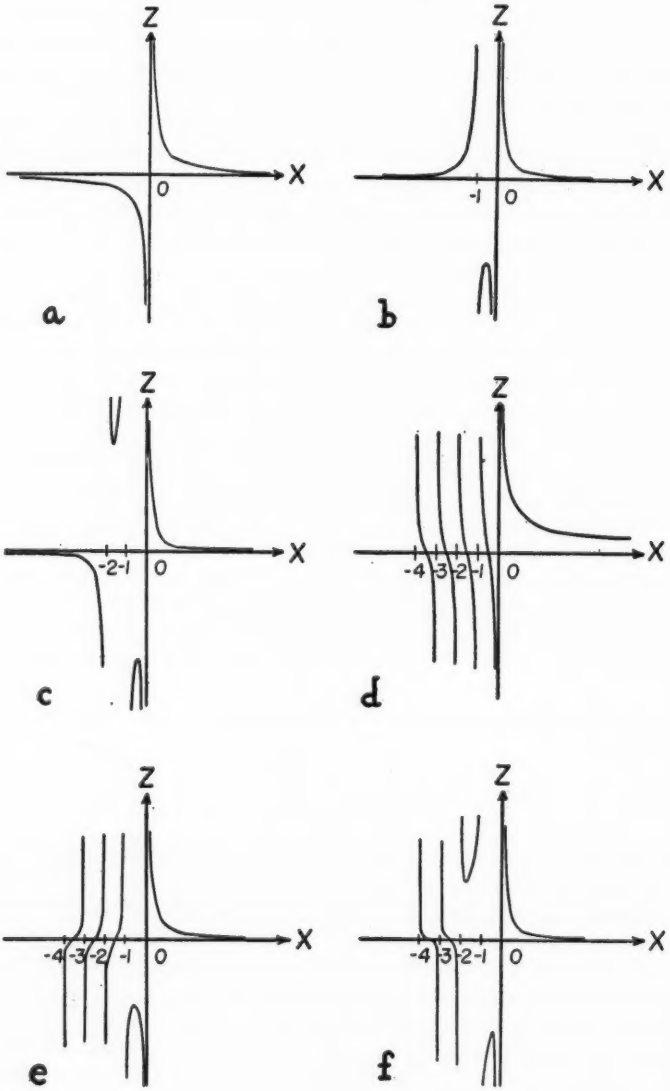


FIG. 3

the line $x=-\frac{1}{2}$. Figure 3(*c*) shows $y=3$, $z=1/x(x+1)(x+2)$, symmetric with respect to the point $(-1, 0)$. Sections where $y=4, 5, \cdots$, continue the trend thus indicated.

If $y=1/2, 3/2, 5/2$ in turn, we obtain the very different traces shown in Figure 3(*d*, *e*, *f*) respectively. Note that all of these have an infinite number of vertical asymptotes, because there is no way for them to get through any of the cell walls to the left of $x=0$. In fact any trace $y=k$, $k$ non-integral, will have an infinite number of asymptotes. If on the other hand $k$ is a positive integer, then the number of vertical asymptotes equals $k$.

Space does not permit analysis of traces $y=-k$, $k$ fractional, which throw further light on the behavior of the surface. Attempts actually to sketch some of the saddle cells require ingenuity in handling perspective and produce some intriguing and often amusing results.

**References**

The writer has been unable to find a pictorial representation of the Beta Function in any of the standard texts, although it would seem that the notion of such a representation, particularly in the first octant, could scarcely be new. Nor is the function generally tabulated. Jahncke-Emde has a sketch of $\Gamma(x)$, and there is a careful sketch of $\Gamma(x)$ for positive $x$ on p. 109, Vol. 2, of Edwards' *A Treatise on the Integral Calculus*, (1922). The extension of the definition to negative values of the arguments of $B(x, y)$ is mentioned often; for example, see Franklin, *A Treatise on Advanced Calculus*, p. 562 (1940). Many books of mathematical tables contain useful tabulations of $\Gamma(x)$.

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## THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH,* College of St. Thomas

The following results of the eleventh annual William Lowell Putnam Mathematical Competition held March 31, 1951, have been determined in accordance with the constitution of the Competition. This Competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of The Mathematical Association of America.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of Cornell University, Ithaca, New York. The members of the

* Director of the Competition.

team were M. Cohen, J. H. Gay, M. Horowitz; to each of these a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were P. Martin, G. Rayna, A. Zemach; to each of these a prize of thirty dollars is awarded.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of The Cooper Union, New York, New York. The members of the team were A. F. Berndt, I. J. Lowe, P. J. Redmond; to each of these a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of the City College of New York, New York, New York. The members of the team were A. Benson, H. Hanisch, H. Widom; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, were A. P. Dempster, University of Toronto; J. B. Herreshoff IV, University of California; Herbert Kranzer, New York University; P. J. Redmond, The Cooper Union; Harold Widom, The City College of New York. Each of these will receive a prize of fifty dollars.

The six succeeding persons ranking highest in the examination, named in alphabetical order, were Paul Cohen, Brooklyn College; Michael Horowitz, Cornell University; P. H. Lord, Princeton University; Arthur Mattuck, Swarthmore College, Gerhard Rayna, Harvard University; Herbert Scarf, Temple University. To each of these a prize of twenty dollars is awarded.

The following teams, named in alphabetical order, won honorable mention: Columbia College, New York, New York, the members of the team being R. Feldman, H. Gordon, C. J. Kaufman; Massachusetts Institute of Technology, Cambridge, Massachusetts, the members of the team being W. L. Baily, L. deBranges III, J. E. Kimber, Jr.; University of British Columbia, Vancouver, British Columbia, the members of the team being E. Critoph, C. A. Swanson, D. A. Trumpler; University of California, Berkeley, California, the members of the team being J. B. Herreshoff IV, R. R. Kissling, Keiichi Nishimura.

Nine individuals were given honorable mention. The names are listed in alphabetical order: Michael Cohen, Cornell University; J. H. Gay, Cornell University; George Gioumousis, Polytechnic Institute of Brooklyn; J. R. Jamieson, University of Toronto; R. Lehman, Stanford University; Hans Stetter, Colorado A & M College; H. F. Trotter, Queen's University; D. A. Trumpler, University of British Columbia; Ariel Zemach, Harvard University.

The following is a list of all colleges and universities which entered teams in the Competition. The list, in alphabetical order, is: Brooklyn College, Carleton College, Carnegie Institute of Technology, City College of New York, College of the Holy Cross, College of St. Thomas, Colorado A & M College, Columbia College, The Cooper Union, Cornell University, Gustavus Adolphus College, Harvard University, Kent State University, Knox College, Laval University,

Loyola College (Montreal), Massachusetts Institute of Technology, McGill University, McMaster University, New York University, Polytechnic Institute of Brooklyn, Princeton University, Queen's University, Reed College, Rutgers University, St. Joseph's College (Connecticut), St. Joseph's College for Women (Brooklyn), Stanford University, State Teachers College (Nebraska), Swarthmore College, University of Arkansas, University of British Columbia, University of Buffalo, University of California (Berkeley), University of Detroit, University of Illinois, University of Miami (Florida), University of Minnesota, University of North Carolina, University of Toronto, Ursinus College, Yale University.

The following additional colleges and universities entered individual contestants only: Brown University, Case Institute of Technology, Central College, Clark University, Clemson Agricultural College, Haverford College, Iowa State College, Loyola University (New Orleans), Pomona College, Purdue University, Temple University, University of Colorado, University of Kentucky, Wayne University.

A total of 209 undergraduates, representing 56 institutions, took part in the Competition.

Participants in the Competition were given the following lists of problems:

PART I. THREE HOURS

*(Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate your answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.)*

*Omit one question. You must indicate which question is omitted.*

1. Show that the determinant:

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}$$

is non-negative, if its elements $a$, $b$, $c$, etc., are real.

2. In the plane, what is the locus of points the sum of the squares of whose distances from $n$ fixed points is a constant? What restrictions, stated in geometric terms, must be put on the constant so that the locus is non-null?

3. Find the sum to infinity of the series:

$$1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \cdots + \frac{(-1)^{n+1}}{3n-2} + \cdots .$$

4. Trace the curve whose equation is:

$$y^4 - x^4 - 96y^2 + 100x^2 = 0.$$

5. Consider in the plane the network of points having integral coordinates. For lines having rational slope show that:
   (a) the line passes through no points of the network or through infinitely many.
   (b) there exists for each line a positive number $d$ having the property that no point of the network, except such as may be on the line, is closer to the line than the distance $d$.
6. Determine the position of a normal chord of a parabola such that it cuts off of the parabola a segment of minimum area.
7. Show that if the series $a_1+a_2+a_3+\cdots+a_n+\cdots$ converges, then the series $a_1+a_2/2+a_3/3+\cdots+a_n/n+\cdots$ converges also.

PART II. THREE HOURS

*(Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate your answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.)*

*Omit one question. You must indicate which question is omitted.*

1. Find the condition that the functions $M(x, y)$ and $N(x, y)$ must satisfy, in order that the differential equation $Mdx+Ndy=0$ shall have an integrating factor of the form $f(xy)$. You may assume that $M$ and $N$ have continuous partial derivatives of all orders.
2. Two functions of $x$ are differentiable, and not identically zero. Find an example of two such functions having the property that the derivative of their quotient is the quotient of their derivatives.
3. Show that if $x$ is positive, then

$$\log_e (1 + 1/x) > 1/(1 + x).$$

4. Investigate in any way which yields significant results, the existence, in the plane, of the configuration consisting of an ellipse simultaneously tangent to four distinct concentric circles.
5. A plane through the center of a torus is tangent to the torus. Prove that the intersection of the plane and the torus consists of two circles.
6. Assuming that all the roots of the cubic equation $x^3+ax^2+bx+c=0$ are real, show that the difference between the greatest and the least roots is not less than $(a^2-3b)^{1/2}$ nor greater than $2(a^2-3b)^{1/2}/3^{1/2}$.
7. Find the volume of the four-dimensional hypersphere $x^2+y^2+z^2+t^2=r^2$, and also the hypervolume of its interior $x^2+y^2+z^2+t^2<r^2$.

## MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, *New York University*

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### A SIMPLIFIED TECHNIQUE FOR A TSCHIRNHAUS TRANSFORMATION

H. S. THURSTON, University of Alabama

**1. Introduction.** If $\rho$ is a root of an irreducible equation $f(x)=0$ with coefficients in a field $\mathfrak{F}$, and $\eta$, an integral rational function of $\rho$, is a primitive number in the field $\mathfrak{F}(\rho)$, then $\rho$ can be expressed as an integral rational function of $\eta$ with coefficients in $\mathfrak{F}$. The classical proof* of this theorem involves the use both of symmetric function theory and the irreducible equation with coefficients in $\mathfrak{F}$ satisfied by $\eta$, and expresses $\rho$ as a rational but nonintegral function of $\eta$. The purpose of this note is to demonstrate a proof of the theorem which expresses $\rho$ directly as a polynomial in $\eta$ with coefficients in $\mathfrak{F}$. The method introduces a function of the coefficients of both $\eta$ and $f(x)$, whose vanishing is a necessary and sufficient condition that $\eta$ be imprimitive.

Although the theory is applicable to any field, it will be understood in what follows that $\mathfrak{F}$ represents the field of rational numbers and that all coefficients are in this field.

**2. The Tschirnhaus transformation.** Let $\rho$ be a root of the irreducible equation $f(x)=\sum_{i=0}^{n} c_i x^i=0$. Let $\eta=\sum_{j=0}^{n-1} a_{1j}\rho^j$ be primitive and express successive powers of $\eta$ in the form

$$\eta^i = \sum_{j=0}^{n-1} a_{ij}\rho^j \qquad (i = 1, 2, \cdots, n-1). \tag{1}$$

Let $B$ denote the value of the determinant of the coefficients of $\rho, \rho^2, \cdots, \rho^{n-1}$, in the system of equations (1). On multiplying these equations respectively by $A_{i1}$, the cofactors of the elements of the first column of the matrix $(a_{ij})$, we obtain an equation

$$\sum_{i=1}^{n-1} A_{i1}\eta^i = A + B\rho \tag{2}$$

where $A$ and $B$ are rational. Since $\eta$, being primitive, satisfies no equation of degree less than $n$ with rational coefficients, $B$ must be different from zero. We can solve (2) for $\rho$ obtaining the desired result.

**3. Imprimitive numbers.** It will be observed that the method of the preceding section determines whether a given number $\eta$ is primitive or not. If $B=0$, $\eta$ satisfies an equation of degree less than $n$ and hence is imprimitive. Conversely, if $\eta$ is imprimitive, satisfying an equation of degree $k<n$, then $B$ must be zero

* See, for instance, C. C. MacDuffee, Introduction to Abstract Algebra (1940), p. 87.

for if not we could solve for $\rho$ as a polynomial in $\eta$ as above. Then $\rho$, being in the field $\mathfrak{F}(\eta)$, will satisfy an equation of degree $k<n$, contrary to the hypothesis. We may therefore state the following theorem.

THEOREM. *A necessary and sufficient condition that $\eta$ be imprimitive is that* $B=0$.

For the reduced cubic $f(x)=x^3+px+q$ and $\eta=a+b\rho+c\rho^2$, we find that $B=b^3+pbc^2+qc^3=c^3f(b/c)$. If $B=0$, then since for $b$ and $c$ rational and $c$ not zero, $f(b/c)$ cannot be zero, we must have $b=c=0$. Hence, in a cubic field, the only imprimitive numbers are the rationals, as is well known from other considerations.

For $n\geqq 4$ and the general number $\eta$, both the computation and the expression for $B$ assume formidable proportions. Some special cases may be of interest. For example, if $f(x)=x^4+px^2+qx+r$ and $\eta=a+b\rho+c\rho^2$, we find that

$$B = b^6 + 2pc^2b^4 + (p^2c^4 - 4rc^4)b^2 - q^2c^6. \tag{3}$$

Since, to be imprimitive, $\eta$ must satisfy an equation of degree two when $n=4$, the coefficient of $\rho^3$ in $\eta^2$ must be zero. This coefficient is $2bc$. The conditions $bc=0$, $B=0$ lead to the conclusion that $\eta$ is imprimitive only when $b=c=0$ or when $b=q=0$.

On the other hand, if $B=0$, and $\eta$ is found to satisfy an equation of degree three but not of degree two, we must conclude that $f(x)$ is not irreducible. For example, if $f(x)=x^4-12x-5$ and $\eta=2+2\rho+\rho^2$, then $B$, given by (3), is zero and we find that $\eta^3-11\eta^2-25\eta+51=0$. But $x^3-11x^2-25x+51$ has factors $x+3$ and $x^2-14x+17$ neither of which vanish for $x=\eta$. Then $f(x)$ is reducible and is, in fact, identical with $(x^2+2x+5)(x^2-2x-1)$.

## A CHARACTERISTIC PROPERTY OF THE CIRCLE IN THE MINKOWSKI PLANE

F. A. VALENTINE, University of California, Los Angeles

**1. Introduction.** Let $M_2$ be a two-dimensional Minkowski space. The unit sphere in $M_2$ is a centrally symmetric convex set having interior points [2, p. 23]. By a circle of radius $r$ and center $a$ we mean the set of points $x\in M_2$ for which the distance function satisfies $\|x, a\|\leqq r$. The circle having the segment $xy$ as a diameter which passes through its center of symmetry is denoted by $K(x,y)$, and the boundary of $K(x, y)$ by $A(x, y)$. A semicircular arc of $A(x, y)$ having $xy$ as a diameter is denoted by $C(x, y)$. If we wish to discuss both semicircular arcs of $A(x, y)$ having $xy$ as a diameter, we do so by means of superscripts. The arc $C(x, y)$ always lies in one of the closed half-planes determined by the line $L(x, y)$ containing $xy$. The reader should realize that semicircular arcs $C(x, y)$ vary in shape in accordance with the direction of $xy$. A point is called an extreme point of $A(x, y)$ if it is *not* an interior point of a line segment of $A(x, y)$.

A set $S \subset M_2$ is said to be arcwise convex if each pair of points $x$ and $y$ in $S$ can be joined by a convex arc $P(x, y) \subset S$. If $P(x, y) = xy$, then $S$ is convex. If $P(x, y)$ is a polygonal arc containing at most two segments, then $S$ is called an $L$ set [4]. We are interested here in investigating the situation when $P(x, y) = C(x, y)$, a semicircular arc. The following theorem has interest from a pedagogical as well as from a mathematical point of view.

THEOREM 1. *A necessary and sufficient condition that a bounded closed set $S \subset M_2$ be a Minkowski circle is that for each pair of points $x \in S$, $y \in S$, there exists a semicircular arc $C(x, y)$ in $S$ having $x$ and $y$ as endpoints.*

*Proof*: In order to prove the sufficiency, the following remark is helpful.

*Remark A. There exist two points $a \in S$, $b \in S$ such that $\|a, b\| =$ diameter of $S$, and such that $a$ and $b$ are extreme points of $A(a, b)$.*

To prove this let the Minkowski diameter of $S$ be $D$, and let $M$ denote the collection of point pairs $[(x, y)]$, $x \in S$, $y \in S$ such that the Minkowski distance $\|x, y\|$ satisfies $\|x, y\| = D$. Let $|x, y|$ denote the Euclidean distance determined by $x$ and $y$. Since $S$ is bounded and closed, there must exist a point pair $(a, b) \in M$ which has a maximum Euclidean distance $|a, b|$ relative to the set $M$. The points $a$ and $b$ must be extreme points of $A(a, b)$; for if this were not so, there would exist two points $u \in C(a, b) \subset S$, $v \in C(a, b) \subset S$ such that $|u, v| > |a, b|$, $\|u, v\| = D$. This proves Remark $A$. The reader should realize that any two points in $S$ which have a maximum Euclidean distance in $S$ do not, in general, have a maximum Minkowski distance in $S$. It is also possible to prove Remark $A$ by an argument which does not refer to the Euclidean distance; however, it is considerably longer, and is therefore not included.

To prove Theorem 1, consider the points $a$ and $b$ in Remark $A$. Let $R(a, b)$ be the closed half-plane containing the arc $C(a, b) \subset S$. Let $L(a)$ and $L(b)$ be two parallel lines of support to $K(a, b)$ at $a$ and $b$ respectively. Since $a$ and $b$ are extreme points of $A(a, b)$, we have either $C(a, b) \cdot L(a) = a$, or $C(a, b) \cdot L(b) = b$, or both. Suppose $C(a, b) \cdot L(a) = a$. Choose a sequence of points $a_i \in C(a, b)$ so that $a_i \to a (a_i \neq a)$ as $i \to \infty$. Let $R(a_i, b)$ be the closed half-plane which is bounded by $L(a_i, b)$, and which contains $a$. Since $a_i \cdot L(a) = 0$, and since the diameter of $S$ is equal to $\|a, b\|$, it follows that the arcs $C(a_i, b) \subset S$ are such that $C(a_i, b) \subset R(a_i, b)$. Hence $C(a_i, b) \to C^*(a, b)$, a semicircular arc in $S$, since $S$ is closed. Moreover, $C(a, b) + C^*(a, b) = A(a, b) \subset S$. Since the diameter of $S$ is equal to $\|a, b\|$, we have shown that the outer boundary of $S$ is the circular circumference $A(a, b)$.

To complete the proof we merely need to show that $S$ is simply connected. Let $\theta$ be the center of $K(a, b)$, and suppose $K(a, b) - S \neq 0$. Choose a point $p \in K(a, b) - S$. Let $L(\theta)$ be a line through $\theta$ which intersects $A(a, b)$ at points $u$ and $v$ at which unique lines of support $T(u)$ and $T(v)$ to $K(a, b)$ exist. Let $R(u, v)$ be a closed half-plane bounded by $L(\theta)$ and containing $p$. Choose $r \in K(a, b) \cdot R(u, v)$ so that a line of support $L(r)$ to $K(a, b)$ at $r$ exists which is parallel to $L(\theta)$. By a continuity argument, there exists a point $c \in pr$ such that the line $L(c)$

through $c$ parallel to $L(\theta)$ has the following property. There exist points $s \in L(c) \cdot A(a, b)$, $t \in L(c) \cdot A(a, b)$ such that $\|m, p\| = \|m, s\| = \|m, t\|$, where $m$ is the midpoint of $ts$. Hence $c \neq p$.

*Case* 1. If $s \in T(u)$, $t \in T(v)$, then the circle $K(s, t)$ has the same diameter as $K(u, v)$. Hence any semicircular arc $C(s, t)$ either contains $p \notin S$, or it intersects the complement of $K(a, b)$. Hence in this case no semicircular arc exists in $S$ joining $s$ and $t$.

*Case* 2. We can suppose, without loss of generality, that $s \notin T(u)$, and that $s$ is nearer to $L(c) \cdot T(u)$ than is $t$. Since $T(u)$ is a unique line of support to $K(a, b)$ at $u$, each line of support to $K(a, b)$ at $s$ must intersect $T(u)$ at a point $q \neq u$. Since $st$ is parallel to $uv$, there is a unique line of support to $K(s, t)$ at $s$, which is moreover parallel to $T(u)$. Hence the semicircular arc $C(s, t)$ not containing $p$ intersects the complement of $K(a, b)$. Hence no semicircular arc in $S$ exists joining $s$ and $t$.

Thus $S$ is simply connected, and, by the previous results, we have shown that $S = K(a, b)$. This completes the sufficiency proof. The proof of the necessity is simple, and will be omitted.

**2. Generalizations.** Theorem 1 does not hold in higher dimensions. For instance, the *solid hemisphere in* $E_3$, the three-dimensional Euclidean space, is a *counterexample*. Each pair of points in the hemisphere can be joined by a semicircular arc lying in the hemisphere. It would be of interest to determine in $E_3$ all of the sets which are arcwise convex via semicircular arcs. If we replace semicircular arcs by hemispherical surfaces, then we can generalize Theorem 1 as follows.

THEOREM 2. *A necessary and sufficient condition that a bounded closed set* $S \subset M_n$ *be a sphere is that for each pair of points* $x \in S$, $y \in S$ *there exists an* $(n-1)$*-dimensional hemispherical surface in* $S$ *having* $xy$ *as a diameter through its center.**

For the sufficiency, let $(a, b)$ be a pair of points $a \in S$, $b \in S$ whose Minkowski distance satisfies $\|a, b\| = D$, where $D$ is the diameter of $S$, such that $(a, b)$ has a maximum Euclidean distance $|a, b|$ among the set of pairs $[(x, y)]$, $x \in S$, $y \in S$ for which $\|x, y\| = D$. Let $P$ be a two-dimensional plane containing the line $L(a, b)$. The two-dimensional plane $P_0$ which is parallel to $P$, and which passes through the center of the unit sphere $U$, intersects $U$ in a two-dimensional Minkowski circle $P_0 \cdot U$. Theorem 2 follows from Theorem 1 since each two-dimensional plane section $P \cdot S$ containing $a$ and $b$ is a circle which is similar to $P_0 \cdot U$, and which has $ab$ as a diameter through its center.

If each pair of points of a set $S$ can be joined by a *circular* arc in $S$, then $S$ belongs to quite a large class of sets. Such a set need not be simply connected. The circumference of a circle is an example. Otto Haupt has investigated sets which he calls "Kreiskonvex" [3]. In such a set $S$ for every *three* points in $S$

* The center of the sphere determined by a hemispherical surface is also called the center of the hemispherical surface.

there exists a circular arc or a segment in $S$ which contains the three points. Haupt has given an interesting classification of such sets. It would also be of interest to classify circularly arcwise convex sets, but this appears to be a difficult problem.

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## CLASSROOM NOTES

Edited by C. B. ALLENDOERFER, University of Washington

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### A THEOREM CONCERNING EXACT DIFFERENTIAL EQUATIONS

R. S. UNDERWOOD, Texas Technological College

The purpose of this note is to call attention to: (a), the simple but unused (at least in most textbooks) generalization of the method of solving exact differential equations in two variables based upon the evaluation of a line integral; and (b) a theorem providing a second method which also applies to the case of $n$ variables, and of which Euler's theorem is a special case.

It is known* that, if

$$Mdx + Ndy = 0 \tag{1}$$

is an exact differential equation whose solution is $F = C$, we may find $F$ by evaluating the line integral $\int_{(a,b)}^{(x,y)}(Mdx+Ndy)$ along any convenient path and from any convenient starting point $(a, b)$. The latter is usually best taken at the origin unless $M$ or $N$ is undefined there; but in any case the effect of a change in $(a, b)$ is merely to change the additive constant in $F$. According to the writer's experience, this method is usually the shortest and simplest one for a given problem, especially when it must be used in generalized form to deal with more than two variables.

* Woods: Advanced Calculus, Revised Edition, page 186.

The generalization is direct and obvious. For example, when three variables are involved we may integrate along the broken line path from (0, 0, 0) to $(x, 0, 0)$ to $(x, y, 0)$ to $(x, y, z)$, or as will be better on occasion, along some different but similar path such as one involving, say, the point $(0, y, 0)$. For any specified number of variables the path of integration may be actually drawn if one uses the writer's "wheel-spoke" system of axes.† Of course the $n$-space concept yields the same result, aside from the picture; but the latter is useful to students in visualizing different paths, each with its separate integration pattern and special order of brevity or difficulty.

Again, we may integrate along the path from the origin to the point, $P(x_1, x_2, \cdots, x_n)$, by use of the parametric equations $y_i = x_i t$ $(i = 1, 2, \cdots, n)$, where the coordinates $x_i$ of $P$ are treated as constants. Evidently the coordinate $y_i$ of a point on the path ranges from 0 to $x_i$ as $t$ ranges from 0 to 1. (On the $n$-axes plane of the writer's system† this path is a straight line.) Then, denoting $\partial F/\partial y_i$ by $F_i$, we have

$$\sum_{i=1}^{n} \int_0^{x_i} F_i(y_1, y_2, \cdots, y_n)\,dy_i = \int_0^1 \sum_{i=1}^{n} F_i(x_1 t, x_2 t, \cdots, x_n t)\,x_i\,dt.$$

The theorem below follows.

THEOREM: *If* $\sum_{i=0}^{n} F_i(x_1, x_2, \cdots, x_n)dx_i$ *is the exact differential of a function* $F$, *and if the* $F_i$ *are defined at the origin, then we may find* $F$, *aside from the arbitrary constant, by replacing* $x_i$ *and* $dx_i$ *by* $x_i t$ *and* $x_i dt$ *respectively, and then evaluating the definite integral in* $t$ *between the limits* 0 *and* 1, *the* $x_i$ *being treated as constants.*

COROLLARY: *If the* $F_i$ *of the theorem are homogeneous, of degree* $k$, *and defined at the origin, then*

$$F = \frac{\sum_{i=1}^{n} x_i F_i}{k+1}.$$

It should be noted that this potent corollary is merely Euler's theorem "in reverse" as compared with its usual presentation. The typical text book approach is illustrated by the following problem: "Verify Euler's theorem for $f(x, y, z) = x^2y + xy^2 + 2xyz$." The student, noting that $(2xy + y^2 + 2yz)x + (x^2 + 2xy + 2xz)y + (2xy)z = 3f$, has duly verified what seems to be an interesting but pointless result. But probably both he and his teacher have failed to notice that if he *starts with* the differential equation $dF \equiv (2xy + y^2 + 2yz)dx + (x^2 + 2xy + 2xz)dy + (2xy)dz = 0$, checks its exactness by the usual test, and observes that the quantities in parentheses are homogeneous and of degree 2, then he need only replace $dx$, $dy$, and $dz$ by $x$, $y$, and $z$ respectively to get $3F$ at once and thus solve the equation. While of course the relative efficiency of the

† Underwood: An Analytic Geometry for $N$ Variables, this MONTHLY, May, 1945.

method increases with the number of variables, in the example cited the result is obtained about as quickly by the line integral method, which yields $F=\int_0^x 0dx+\int_0^y(x^2+2xy)dy$ (with $x$ constant) $+\int_0^z 2xydz$ ($x$ and $y$ constant) $=x^2y+xy^2+2xyz$.

*Editorial Note.* A general form of Underwood's theorem has been proved by H. Cartan in connection with exterior differential forms. For the simple case of $Mdx+Ndy$, let $F(x, y)=\int_0^1[M(xt, yt)x+N(xt, yt)y]dt$. Then Cartan's result is:

$$\frac{\partial F}{\partial x} = M(x, y) + \int_0^1 \left[\frac{\partial N(xt, yt)}{\partial(xt)} - \frac{\partial M(xt, yt)}{\partial(yt)}\right] ytdt;$$

$$\frac{\partial F}{\partial y} = N(x, y) - \int_0^1 \left[\frac{\partial N(xt, yt)}{\partial(xt)} - \frac{\partial M(xt, yt)}{\partial(yt)}\right] xtdt.$$

This proves Underwood's theorem without resorting to other theorems from advanced calculus. It also proves that $Mdx+Ndy$ is exact whenever

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0;$$

and thus at a single stroke one can prove all the important facts about $Mdx+Ndy$.

This method can be extended to more dimensions and to differential forms of higher degree. When applied to the form: $Pdx+Qdy+Rdz$, it proves the theorem: "If a vector has zero curl, then it is a gradient." Application of a similar technique proves the theorem: "If a vector has zero divergence, then it is the curl of another vector." The trick is to set:

$$X(x, y, z) = \int_0^1 [Q(xt, yt, zt)zt - R(xt, yt, zt)yt]dt$$

$$Y(x, y, z) = \int_0^1 [R(xt, yt, zt)xt - P(xt, yt, zt)zt]dt$$

$$Z(x, y, z) = \int_0^1 [P(xt, yt, zt)yt - Q(xt, yt, zt)xt]dt$$

Then curl $(X, Y, Z)=(P, Q, R)$ provided that div $(P, Q, R)=0$.

## THE BETA FUNCTION

M. R. SPIEGEL, Rensselaer Polytechnic Institute

In all of the books on Advanced Calculus which the author has seen, the evaluation of the Beta function

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \qquad m, n > 0 \tag{1}$$

is performed with the aid of a suitable double integral. If the Laplace transform is used, this evaluation is immediate.

Consider

$$F(t, m, n) = \int_0^t x^{m-1}(t - x)^{n-1}dx. \tag{2}$$

Taking the Laplace transform with respect to $t$ we have

$$\mathcal{L}_t F(t, m, n) = \mathcal{L}_t(t^{m-1})\mathcal{L}_t(t^{n-1}) \tag{3}$$

by the convolution theorem.

Since

$$\mathcal{L}_t(t^p) = \frac{\Gamma(p + 1)}{s^{p+1}}$$

(3) becomes

$$\mathcal{L}_t F(t, m, n) = \frac{\Gamma(m)\Gamma(n)}{s^{m+n}}. \tag{4}$$

Inverting now with respect to $t$ we have

$$F(t, m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m + n)} t^{m+n-1}$$

since

$$\mathcal{L}^{-1}\left(\frac{1}{s^p}\right) = \frac{t^{p-1}}{\Gamma(p)}.$$

If, now, we put $t=1$ the result (1) follows.

## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

*Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.*

### PROBLEMS FOR SOLUTION

E 976. *Proposed by R. D. Stalley, University of Arizona*

What is the least area of a right triangle which can be placed on a rectangular coordinate system so that no side is parallel to a coordinate axis and so that the coordinates of its vertices and the lengths of its sides are integral?

E 977. *Proposed by José Gallego-Diaz, Madrid, Spain*

Let $OA$ be a radius of a circle with center $O$. Produce $OA$ its own length to $B$. Through $A$ draw a horizontal line and on it mark off, in each direction from $A$, $AM=AN=OA$. If $OB$ and $MN$ are taken as the equal conjugate diameters of an ellipse, find the loci of the vertices and the foci of this ellipse as the radius $OA$ varies. Also find the envelope of the major axis of the ellipse as $OA$ varies.

E 978. *Proposed by J. M. Kingston, University of Washington*

If $[x]$ denotes the integral part of $x$ and if $j$ is any positive integer, show that

$$\sum_{i=0}^{[j/3]} [(j-3i)/2] = [(j^2+2j+4)/12].$$

E 979. *Proposed by E. S. Keeping, University of Alberta*

A student is asked on a test to match $n$ given historical events against $n$ different given dates, assigning a date to each event. He is sure of $k$ dates, but is completely ignorant of the others and assigns them at random. Show that a fair method of scoring would be to deduct 1 from the number of correct matches.

### SOLUTIONS

#### A Quadratic Diophantine Equation

E 941 [1950, 686]. *Proposed by P. A. Piza, San Juan, Puerto Rico*

Find positive integers $a$, $b$, $c$ such that $a^2+b^2+3c^2=(a+b+c)^2$.

*Solution by E. P. Starke, Rutgers University.* Given any two relatively prime positive integers $x$, $y$, such that $x<y<2x$, and an arbitrary positive integer $k$, then the complete solution is

$$c = kxy, \qquad a = kx(2x-y), \qquad b = ky(y-x), \tag{1}$$

where the values of $a$ and $b$ may be interchanged.

That these numbers satisfy is seen immediately upon substituting them in the given equation. Conversely, if $a$, $b$, $c$ are positive integers which satisfy the given equation, or its equivalent

$$2c^2 = (c + a)(c + b), \tag{2}$$

then there exist integers $x$, $y$ such that (1) is satisfied. To show this, suppose that $a$, $b$, $c$ have a greatest common divisor $k$. Then $a' = a/k$, $b' = b/k$, $c' = c/k$ have no common factor and satisfy

$$2c'^2 = (c' + a')(c' + b'). \tag{2'}$$

Then $c'+a'$ and $c'+b'$ are relatively prime and, therefore, from (2'),

$$c' + a' = 2x^2, \qquad c' + b' = y^2, \qquad c' = xy, \tag{3}$$

or the same with $a'$ and $b'$ interchanged. Now, from (2') and (3), we have $c' < y^2 < 2c'$, whence $x < y < 2x$, which establishes the proposition.

Also solved by Furio Alberti and N. C. Scholomiti (jointly), R. V. Andree, Norman Anning, W. E. Baxter, Alan Berndt, Daniel Block, J. L. Botsford, W. G. Brady, D. H. Browne, R. L. Caskey, P. L. Chessin, F. L. Dennis, F. J. Duarte, C. C. Foster, Jr., R. E. Gettig, A. L. Gilmore, Jr., Vern Hoggatt, M. S. Klamkin, Joseph Klein, Sam Kravitz, Max LeLeiko, Roger Lessard, Theodore Lindquist, David Mandelbaum, D. V. Mardle and R. J. Taylor (jointly), Fred Marer, D. C. B. Marsh, Eric Mickalup, Leo Moser, Prasert Na Nagara, C. S. Ogilvy, M. Paul, L. A. Ringenberg, J. E. Sanders, C. M. Sandwick, Jr., J. W. Sawyer, O. E. Stanaitis, D. Strebe, Elijah Swift, R. S. Underwood, N. D. Vlachos, J. E. Weidlich, and the proposer. Many of these solutions were incomplete.

**Maximinimal Configuration of Five Points on a Sphere**

E 946 [1951, 36]. *Proposed by C. S. Ogilvy, Columbia University*

It is well known that the distribution of six points on the surface of a given sphere which makes the least distance between any pair a maximum is that of the vertices of the regular octahedron. What is the corresponding distribution for five points?

*Solution by Leo Moser, Texas Technological College.* There is no unique distribution which maximizes the minimum distance between five points on a sphere. A set of maximinimal configurations is given by taking two points at the north and south poles and the remaining three on the equator, subject only to the condition that the distance between any two of these be at least 90°.

To show that we cannot have each distance greater than 90° note that if this be the case then two points will eliminate the possibility of the remaining three lying in two hemispheres whose poles are more than 90° apart. Thus these three points must lie in a lune of angle less than 90°, or at least two of them must lie

in or on one half this lune, *i.e.*, in or on a triangle having two sides each equal to 90° and the third side less than 90°. But this is impossible.

Also solved by the proposer.

It is interesting that five points can be spaced no farther apart than six.

### A Locus Problem

E 947 [1951, 36]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find the locus of a point in the plane of a given triangle such that the pedal triangle of its isogonal conjugate is right angled.

I. *Solution by K. W. Crain, Purdue University.* Let the given triangle be $A_1A_2A_3$, the considered point $P'$, and its isogonal conjugate $P$, whose pedal triangle is $P_1P_2P_3$. Since $A_iP'$ is perpendicular to $P_jP_k$ (*cf.* Johnson, Modern Geometry, sec. 237, page 156), the requirement that the pedal triangle be right angled at $P_k$ is equivalent to making the angle $A_iP'A_j=90°$. Thus the locus of $P'$ consists of the three circles having the sides of the given triangle as diameters.

II. *Solution by Roscoe Woods, State University of Iowa.* In this solution the pedal triangle of a point $P$ in the plane of the given triangle $ABC$ is defined as the triangle $LMN$, where $L$, $M$, $N$ are, respectively, the traces of $AP$, $BP$, $CP$ on the sides $BC$, $CA$, $AB$ (*cf.* Altshiller-Court, College Geometry, page 128).

If the trilinear (normal) coordinates of $P$ are $(x, y, z)$, referred to triangle $ABC$ as reference triangle, the coordinates of $P'$, the isogonal conjugate of $P$, are $(yz, zx, xy)$. The coordinates of the vertices of the pedal triangle $L'M'N'$ of $P'$ are readily found to be: $L'(0, z, y)$, $M'(z, 0, x)$, $N'(y, x, 0)$. The condition that triangle $L'M'N'$ be right angled at vertex $L'$, say, is found to be

$$x^2 - y^2 - z^2 - 2yz\cos A = 0. \tag{1}$$

Hence if the point $P$ lies on the conic whose equation is (1), the pedal triangle $L'M'N'$ is right angled at the vertex $L'$.

The locus of (1) is never an ellipse. This conic cuts the side $AC$ in the same points as do the internal and external bisectors of angle $B$. A similar statement holds for side $AB$. It is readily seen that the points $(a, -b, c)$ and $(a, b, -c)$ lie on the conic. These points are the intersections of the tangents to the circle $ABC$ at $B$ and $C$ with the tangent at $A$. The center of this conic is at the point $(-\sin^2 A, \cos C, \cos B)$, which lies on the altitude $h_a$.

The locus of (1) is a parabola when $A=90°$. Its focus is the vertex $A$, its directrix is the side $BC$, and its point of contact with the ideal line is the point $(-a, b, c)$. The vertex of the parabola is the midpoint $(a, b, c)$ of $h_a$, and is the symmedian point of the triangle $ABC$.

The locus of (1) is a rectangular hyperbola when $A=45°$ or $135°$. The centers of these two hyperbolas lie on the altitude $h_a$ and have the coordinates $(\pm 1, 2\cos C, 2\cos B)$.

Also solved by A. Sisk (as in II) and the proposer (as in I).

**The Inverse of a Certain Matrix**

E 948 [1951, 36]. *Proposed by Roy Dubisch, Fresno State College*

Find the inverse of the general even ordered skew-symmetric matrix all of whose elements above the principal diagonal are equal to 1.

I. *Solution by C. S. Ogilvy, Columbia University.* If a skew-symmetric matrix $A$ has elements $a_{ij}=1$ for all $j>i$, then $a_{ij}=-1$ for all $j<i$, and $a_{ii}=0$. Form the matrix $C=AB$, where $B$ has elements $b_{ij}$ such that $b_{ij}=(-1)^{i+j}$ for all $j>i$, $b_{ij}=(-1)^{i+j-1}$ for all $j<i$, and $b_{ii}=0$. Then $C$ has elements $c_{ij}$ such that

$$c_{ij} = (n/2-1)(1) + (n/2-1)(-1) + (2)(0) = 0, \qquad \text{all } i \neq j,$$

$$c_{ii} = (n/2)(1) + (n/2-1)(-1) + (1)(0) = 1.$$

That is, $C=I$, or $B=A^{-1}$.

II. *Solution by D. R. Morrison, Tulane University.* Let $2n$ be the order of the given matrix $A$ and consider the matrix $X$, also of order $2n$,

$$X = \begin{bmatrix} 0 & 1 & 0 & 0\cdots 0 \\ 0 & 0 & 1 & 0\cdots 0 \\ 0 & 0 & 0 & 1\cdots 0 \\ \cdot & \cdot & \cdot & \cdots\cdot \\ 0 & 0 & 0 & 0\cdots 1 \\ -1 & 0 & 0 & 0\cdots 0 \end{bmatrix}.$$

One may easily verify that

$$X^{2n} = -I,$$

where $I$ is the identity matrix of order $2n$, and

$$A = f(X) = X + X^2 + X^3 + \cdots + X^{2n-1}.$$

It follows that

$$(I-X)f(X) = X - X^{2n} = I + X.$$

Now $(I-X)$ has determinant 2, so it has an inverse, and

$$f(X) = (I+X)(I-X)^{-1}.$$

Also $(I+X)$ has determinant 2, so it has an inverse, and

$$[f(X)]^{-1} = (I-X)(I+X)^{-1} = f(-X).$$

Thus

$$A^{-1} = f(-X) = -X + X^2 - X^3 + \cdots - X^{2n-1}.$$

This is the skew-symmetric matrix of order $2n$ whose elements above the main

diagonal are alternately $+1$ and $-1$ in each row and column, with a $-1$ in the upper right-hand corner.

Also solved by E. V. Haynsworth, A. E. Livingston, Leo Moser, W. V. Parker, R. E. Shear, W. L. Stamey, O. E. Stanaitis, and the proposer.

Several solvers computed the inverse by the standard formula $A^{-1} = (\text{adj } A)/|A|$, or by the method of elementary transformations as explained, *e.g.*, in A. A. Albert, *A rule for computing the inverse of a matrix*, this MONTHLY, 1941, pp. 198–99. One solver found the inverse by the method given by R. V. Andree in *Computation of the inverse of a matrix*, this MONTHLY, 1951, pp. 87–92.

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## ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

*Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.*

### PROBLEMS FOR SOLUTION

4448. *Proposed by Jekuthiel Ginsburg, Yeshiva University, New York City*

Solve in positive integers

$$z^4 = \frac{ax^2 + by^2}{a + b},$$

where $a$ and $b$ are given integers.

4449. *Proposed by D. J. Newman, New York University*

The series $\sum z^n/n$ converges everywhere on the unit circle except at $z = +1$. Can $\pm$ signs be introduced so that $\sum \pm z^n/n$ will converge everywhere on the unit circle with no exceptions?

4450. *Proposed by R. M. Redheffer, University of California at Los Angeles*

Let $g(x)$ be a polynomial of degree $\leqq 2n+1$. Show that a necessary and sufficient condition that the integral

$$\int e^{x^2} g(x)\,dx$$

is an elementary function is the vanishing of the determinant

$$\begin{vmatrix} g(0) & 1 & 0 & 0\cdots & 0 & 0 \\ g^{(2)}(0) & 4 & 1 & 0\cdots & 0 & 0 \\ g^{(4)}(0) & 0 & 8 & 1\cdots & 0 & 0 \\ g^{(6)}(0) & 0 & 0 & 12\cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g^{(2n-2)}(0) & 0 & 0 & 0\cdots & 4n-4 & 1 \\ g^{(2n)}(0) & 0 & 0 & 0\cdots & 0 & 4n \end{vmatrix}.$$

4451. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Let

$$f(z) = \prod_{j=1}^{n} (z - z_j), \qquad |z_j| = 1.$$

Prove that one can connect the origin with a point of the unit circle, so that everywhere on this path, except at the origin, $|f(z)| < 1$.

4452. *Proposed by Leo Moser, Texas Technological College*

Prove that

$$\sum_{r=1}^{n} \mu(r)\left[\frac{n}{r}\right]\{3 - 2\omega(r)\}^2 = 9 + 8\{\pi_1^*(n) + \pi_2^*(n)\},$$

where $\mu(r)$ is the Möbius function, $[n/r]$ denotes the largest integer not exceeding $n/r$, $\omega(r)$ is the number of distinct prime divisors of $r$ (i.e., $\omega(1)=0$, $\omega(p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k})=k$), and $\Pi_k^*(n)$ is the number of integers $t$ not exceeding $n$ for which $\omega(t)=k$.

## SOLUTIONS

### Sequence of Integers No One of which Divides Another

4363 [1949, 557]. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Let $a_1<a_2<\cdots$ be an infinite sequence of positive upper density (i.e., $\underline{\lim}\, a_k/k < \infty$). Then there exists an infinite subsequence such that no element divides another. In fact, there exists an infinite subsequence $a_{i_1}, a_{i_2}, \cdots$ such that $\sum 1/a_{i_k} = \infty$ and no $a_{i_k}$ divides any other.

*Solution by the Proposer.* First we show that there exists an infinite sequence $a_{i_1}<a_{i_2}<\cdots$ such that no one divides another. Put $\underline{\lim}\, a_k/k = 1/\alpha$, $0<\alpha\leqq 1$. Then clearly $\alpha$ is the upper density of the $a$'s. Let now $a_{i_1}$ be the least $a>4/\alpha$, $a_{i_2}$ the least $a>2a_{i_1}$ which is not a multiple of $a_{i_1}$, $a_{i_3}$ the least $a>2a_{i_2}$ which is not a multiple of $a_{i_1}$ or $a_{i_2}$, and so on. Suppose, if possible, that this

process stops after $k$ steps. Then every sufficiently large $a$ is a multiple of one of the numbers $a_{i_1}, a_{i_2}, \cdots, a_{i_k}$. By definition $a_{i_j} > 2^j \cdot 4/\alpha$. Thus the density of the integers which are multiples of one of the numbers $a_{i_1}, a_{i_2}, \cdots, a_{i_k}$ is not greater than

$$\frac{1}{a_{i_1}} + \frac{1}{a_{i_2}} + \cdots + \frac{1}{a_{i_k}} < \frac{\alpha}{4} \sum_{j=1}^{\infty} \frac{1}{2^j} = \frac{\alpha}{2},$$

or the upper density of the $a$'s is less than $\alpha/2$, a contradiction.

To prove the stronger result, denote by $d_x$ the density of integers which have a divisor between $x$ and $2x$. It has been shown that* as $x \to \infty$, $\lim d_x = 0$. Now we define a sequence $n_1 < n_2 < \cdots$ as follows:

(1) $$d_{n_r} < \alpha/10^r,$$

(2) the number of $a$'s in $(n_r, 2n_r)$ is $> \frac{1}{2}\alpha n_r$.

(This condition can be satisfied since the upper density of the $a$'s is $\alpha$.) Define our subsequence of the $a$'s as follows: Consider the $a$'s in $(n_r, 2n_r)$ which are not multiples of any integer in $(n_j, 2n_j)$, $j = 1, 2, \cdots, r-1$. Thus we obtain the sequence $a_{i_1}, a_{i_2}, \cdots$. Clearly no one of these divides any other. Also the number of $a_{i_k}$'s in $(n_r, 2n_r)$ is not less than

$$\frac{\alpha}{2} n_r - \sum \frac{\alpha n_r}{10^j} > \frac{\alpha}{4} n_r.$$

(The number of the $a_{i_k}$'s in $(n_r, 2n_r)$ is greater than the number of $a$'s in $(n_r, 2n_r)$ minus the number of integers in $(n_r, 2n_r)$ which have a divisor in $(n_j, 2n_j)$, $j = 1, 2, \cdots, r-1$). Hence the upper density of the $a$'s is $>0$ and thus $\sum 1/a_{i_k} = \infty$.

It is not hard to see that these results are the best possible. In other words if $f(x) \to \infty$ arbitrarily slowly there exists a sequence $a_1 < a_2 < \cdots$ such that for all $x$ the number of $a_i$'s $\leqq x$ exceeds $x/f(x)$, but there does not exist an infinite subsequence $a_{i_1} < a_{i_2} < \cdots$ no one of which divides another.

**Regular Polygon**

4364 [1949, 557]. *Proposed by Joseph Rosenbaum, The Milford School, Connecticut*

On the sides $A_iA_{i+1}$ of an $n$-gon $A_1A_2 \cdots A_n$ as bases, isosceles triangles $A_iA_{i+1}B_i$ are constructed, either all exteriorly or all interiorly, with the vertex angle $B_i = 360°/n$. Prove

(a) If $A_1A_2 \cdots A_n$ is a projection of a regular $n$-gon, then $B_1B_2 \cdots B_n$ is regular.

(b) The problem of locating the points $A_i$ when the points $B_i$ are given is a porism.

* Erdös, Note on sequences of integers no one of which divides the other, *London, Math. Soc. Journal*, 1935.

*Solution by J. B. Kelly, Institute for Advanced Study.* (a) Let the vertices $A_1, \cdots, A_n$ lie in the complex plane with coördinates $z_1, \cdots, z_n$. Let the coördinates of $B_1, \cdots, B_n$ be $z_1', \cdots, z_n'$. Let $\phi = 2\pi/n$. Then clearly

$$z_{i+1} - z_i' = \omega(z_i' - z_i), \qquad i = 1, \cdots, n,$$

where $\omega = e^{i\phi}$. From this it follows that

$$z_i' = \frac{-\omega}{1-\omega} z_i + \frac{z_{i+1}}{1-\omega}, \qquad i = 1, \cdots, n, \tag{1}$$

in which $n+1$ is to be identified with 1, and so on.

If the $n$-gon $A_1A_2 \cdots A_n$ is the projection of $C_1 \cdots C_n$, where the coördinates of the vertices $C_1, \cdots, C_n$ in their plane are $z_1^*, \cdots, z_n^*$, then there exist constants $p, q, r$ such that

$$z_i = pz_i^* + q\bar{z}_i^* + r, \tag{2}$$

If $C_1 \cdots C_n$ is a regular $n$-gon, then either

$$z_{i+1}^* - z_i^* = \omega(z_i^* - z_{i-1}^*) \tag{3}$$

or

$$z_{i+1}^* - z_i^* = \bar{\omega}(z_i^* - z_{i-1}^*). \tag{4}$$

Substituting (2) and (3) in (1) we find, after some simplifications,

$$z_{i+1}' - z_i' = \bar{\omega}(z_i' - z_{i-1}'), \tag{5}$$

while, substituting (2) and (4) in (1) gives the same as (5) except that $\bar{\omega}$ is replaced by $\omega$. In either case $B_1 \cdots B_n$ is regular.

(b) The determinant of the system of linear equations (1) is

$$(-1)^n(\omega^n - 1)/(1-\omega)^n = 0.$$

Hence the problem of locating the $A$'s when the $B$'s are given has an infinite number of solutions, if any, and is a porism.

In (a) and (b) it has been assumed that $A_iA_{i+1}B_i$ was constructed exterior to the polygon, and that the vertices $A_1, \cdots, A_n$ were taken clockwise.

**Limit of a Ratio of Sums**

4378 [1950, 41]. *Proposed by H. W. Smith, Oklahoma Agricultural and Mechanical College*

Determine

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} {}_nC_k/(2k+1)}{\sum_{k=0}^{n} {}_{n+1}C_k/(2k+1)},$$

where ${}_nC_k$ are the binomial coefficients.

*Solution by Joshua Barlaz, Rutgers University.* Let

$$I_n = \int_0^1 (1 + x^2)^n dx = \sum_{k=0}^{n} {}_nC_k/(2k + 1).$$

We shall prove

$$2^n/n < I_n < 2^n/(n - 1) \tag{1}$$

for $n \geqq 3$. By integration by parts we obtain

$$I_{n+1} = \frac{(2n + 2)I_n}{2n + 3} + \frac{2^{n+1}}{2n + 3}.$$

Now if (1) is true, we can write

$$\frac{(2n + 2)2^n}{n(2n + 3)} + \frac{2^{n+1}}{2n + 3} < \frac{(2n + 2)I_n}{2n + 3} + \frac{2^{n+1}}{2n + 3} < \frac{(2n + 2)2^n}{(n - 1)(2n + 3)} + \frac{2^{n+1}}{2n + 3},$$

$$\frac{2^{n+1}}{n + 1} < \frac{2^{n+1}(2n + 1)}{n(2n + 3)} < I_{n+1} < \frac{2^{n+1}(2n)}{(n - 1)(2n + 3)} \leqq \frac{2^{n+1}}{n}$$

provided $n \geqq 3$. But (1) is true for $n = 3$, whence by induction (1) is true for all $n \geqq 3$. From (1) it follows that $I_n$ is asymptotic to $2^n/n$ and therefore $I_n/I_{n+1} \to \frac{1}{2}$ as $n \to \infty$ which is the desired result.

Also solved by D. F. Barrow, N. J. Fine, M. S. Klamkin, J. G. Millar, C. E. Reid, O. E. Stanaitis, Robert Steinberg, H. E. Stelson, and the proposer.

*Editorial Note.* Klamkin shows that the desired result follows directly from Polya and Szegö, *Aufgaben und Lehrsätze*, vol. 1, p. 78, theorem 199. By the same argument it is true that

$$\lim_{n \to \infty} I_{r,n}/I_{r,n+1} = \tfrac{1}{2}, \qquad I_{r,n} = \sum_{k=0}^{n} {}_nC_k/(rk + 1).$$

**An Infinite Series of Successive Integrals**

4382 [1950, 119]. *Proposed by E. P. Starke, Rutgers University*

Let $f_1(x)$ be Riemann integrable in the interval $0 \leqq x \leqq M$ and let

$$f_{n+1}(x) = \int_0^x f_n(x)dx, \qquad n = 1, 2, \cdots.$$

Show that

$$\phi(x) = \sum_{n=1}^{\infty} f_n(x)$$

is defined and continuous in the interval except perhaps at discontinuities of

$f_1(x)$, and find a simple expression for $\phi(x)$.

I. *Solution by N. J. Fine, University of Pennsylvania.* By repeated integration by parts, it is easy to see that

$$\int_0^x e^{-t} f_1(t)dt = e^{-x}\{f_2(x) + \cdots + f_n(x)\} + \int_0^x e^{-t} f_n(t)dt. \tag{1}$$

Now $f_2(t)$ is continuous, hence bounded for $0 \leqq t \leqq M$. If $K$ is the bound, an easy induction shows that

$$|f_n(t)| \leqq K \frac{t^{n-2}}{(n-2)!}.$$

Hence the integral in the right member of (1) tends to zero, so that

$$\phi(x) = f_1(x) + e^x \int_0^x e^{-t} f_1(t)dt.$$

The result could also have been obtained by observing that, formally, $\phi(x) - f_1(x)$ satisfies the differential equation

$$u' = f_1(x) + u.$$

The justification of this method is also quite easy.

II. *Solution by R. P. Agnew, Cornell University.* With the weaker hypothesis of Lebesgue integrability, the familiar formula

$$f_n(x) = \int_0^x \frac{(x-t)^{n-2}}{(n-2)!} f_1(t)dt \qquad n = 2, 3, \cdots,$$

is easily proved by induction, the only non-trivial part of the proof being inversion of order of integration which is justified by the Fubini theorem. The Lebesgue criterion of dominated convergence for termwise integration of series then gives

$$g(x) \equiv \sum_{n=2}^{\infty} f_n(x) = \int_0^x e^{x-t} f_1(t)dt.$$

Since $g(x)$, being the product of $e^x$ and another continuous function, is continuous, the function $\phi(x) = f_1(x) + g(x)$ is continuous for precisely those values of $x$ for which $f_1(x)$ is continuous.

Also solved by W. G. Brady, R. C. Buck, J. R. Duffett, R. M. Gordon, Sigurdur Helgason, Melvin Henriksen, M. S. Klamkin, A. D. Martin, J. G. Millar, H. L. Reed, O. E. Stanaitis, and Donald Thompson, Jr.

## RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

*All books for review should be sent directly to the editor of this department, American Mathematical Monthly 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.*

*The Main Stream of Mathematics*. By Edna E. Kramer. Oxford University Press, New York. 1951. 321 pages.

The book was written, primarily as a popular exposition of the fundamental concepts of mathematics, their historical development, and their significance in the modern world. It will, however, prove of equal value to the student, and even to the teacher of mathematics. In broad sweeps it delineates the main features of the entire structure of mathematics, from the primitive attempts at counting to the emergence of transfinite numbers, from early Egyptian geometry to the modern theories of non-Euclidean and multi-dimensional geometry and the structure of relativistic space.

While the book does not purport to be a connected history of mathematics, the author manages to impress the reader with the essential continuity of mathematical ideas. She has quite an original way of passing, almost imperceptively, from a discussion of early mathematical concepts to their embodiment in some of the latest developments. In the first chapter, for example, she takes up the evolution of counting and the rise of systems of enumeration in the ancient world and immediately goes on to show the close connection of those early accomplishments with the new science of Cybernetics and the modern computing machine. Similarly, in the third chapter, *The Binding Energy of Thought*, she starts with a discussion of the meaning of algebra and the introduction of literal symbols by the French mathematician, Vieta. However, for illustrations of the use of algebraic symbols the author draws upon such topics, as: *Boolean Algebra*, Einstein's *Mass Energy Formula*, *Cosmic Rays*, etc. Finally, she shows how algebraic symbolism can be used in art by citing Professor Birkhoff's formula for aesthetic measure.

Among the best chapters of the book are: *Science and the Sweepstakes*, in which the author starts with the elementary illustration of tossing a coin and leads the reader through various applications of probability and statistics to the most advanced applications by Von Neuman and Morgenstern in Theory of Games; *From Alice to Einstein*, in which the modern concepts of the nature of mathematics are introduced by means of showing how geometry has changed under the influence of Hilbert and the non-Euclidean geometers; *The Paradise of Mathematicians*, which indicates the close connection between Zeno's paradoxes and the modern notion of the continuum as influenced by Cantor's transfinite numbers.

One may question, of course, how much of a real understanding the non-initiated—that mythical layman—for whom such books presumably are writ-

ten, can gain of such abstract concepts as: dense in itself, continuum, and so forth, by a concise definition, even if it is followed by an illustration. Every teacher knows how much even the advanced student has to struggle with these ideas when he meets with them for the first time. Such difficulties must arise in every attempt to make the abstractions of modern mathematics understandable to the non mathematician. However, the student of mathematics will read this book with profit and pleasure. He will delight in the warm personal touch with which the author introduces her short biographical sketches. The teacher of mathematics will find this book a most valuable addition to his reference material, to be assigned to his students for outside reading.

A. D. FLESHLER

*Mathematics, Queen and Servant of Science.* By E. T. Bell. McGraw-Hill Book Company, Inc., New York. 1951. 15+437 pages, $5.00.

This book is a thorough revision and an amplified integration of two popular volumes of mathematics, *The Queen of the Sciences*, 1931 and its sequel, *The Handmaiden of the Sciences*, 1937 by the author. Here is a fascinating account of selected topics from the huge accumulation available in the developments in pure and applied mathematics from the geometry of Euclid to the most recent findings in mathematical physics. Along with discussions in logic, rings, groups, transformations, topology, continuity, geometry, calculuses, infinity, probability, and others, the author introduces subjects such as the unified field theory (Einstein), Warings problem (Niven), Mersenne numbers (Lehmer), three-valued logic and quantum mechanics (Reichenbach and Post). Informally written and characterized by the author's provocative and stimulating style, mathematical ideas are presented and evaluated to suggest that mathematics is vigorously alive, is still growing, and is indispensable to an understanding of some sciences and technologies and for a deeper appreciation of the philosophy of science. This theme is the greatest virtue of the book.

Beyond this, the author's aims are modest. The sketch is not intended to be a substitute for a textbook in any of his chosen subjects. Some of these are presented in profuse detail. As an example, the chapter on "The Art Of Abstraction" is replete with postulates, terminology and notations from the logic of Tarski, through lattice theory to linear algebras including several other "structures" and abstractions. A splendid feature and a saving grace is the italic print of important terms. The reader may take what he likes and he must explore fuller accounts if he desires to know more. An appeal is made to the intuition of the reader in other chapters. This is inevitable because of the nature and scope of the topics, although (topology and integration for example) could be expanded or clarified. Again, although the sketch makes seemingly countless references to outstanding men and to hundreds of terms, it is not a history of mathematics. Only 202 of some 8000 mathematicians and physicists are actually mentioned.

This book should be valuable to many readers. The non-mathematician will enjoy the pervading spirit and vitality of living mathematics. The student will find new fields for fuller development by himself. The mathematical amateur will gain background and learn to appreciate techniques in research. The teacher will find stimulating material and methods for presentation. The philosopher and the specialist will find many topics of difference and disagreement in the authors overemphasis of some items and the lack of significant development of others.

F. G. GRAFF

Στοιχεια Θεωρητικης Γεωμετριας (*Elements of Theoretical Geometry*): 3 Vols. By N. Sakellariou. Athens, Pountza Book Company, 1950. Vol. I, 224 pages; Vol. II, 208 pages; Vol. III, 208 pages. 20,000 drachmas (about $1.35) per volume.

*Elements of Theoretical Geometry* is a comprehensive text-book in the field of Euclidean geometry, designed for general use in the upper grades of the Greek public gymnasia and for college preparation in the private schools.

Volume One is confined to the general topics contained in Books One and Two of Euclid's *Elements*. In addition to the ancient elementary concepts of Euclidean geometry, some advanced notions and modern extensions are introduced in the sections to which they apply. Examples of such material are the following: the orthocenter of a triangle; inscribed and circumscribed quadrilaterals; classical theorems concerning the Steiner-Miquel point, the Euler line, the nine-point circle, the Simson line, and the Brocard point.

Volume Two, continuing with the substance of Euclid's Books Three and Four, develops most completely the concepts of modern Euclidean geometry. It comprises such topics as the power of a point with respect to a circle, transversals, homothetic figures, poles and polars, inverses; the famous classical theorems of Clairaut, Stuart, Salmon, Compaignon, and Gergonne; the Lemoine point, antiparallels, transpolar triangles, the circle of Apollonius, the circles of Lemoine, Brocard, Tucker.

Volume Three, corresponding to Euclid's Books Five, Six and Seven, is a rather exhaustive work in the Euclidean geometry of three-space. A special distinction is the application to three-space of the concepts of the power of a point, homothetic figures, poles and polars, and so on.

The author maintains the same plan of presentation and technique throughout the three volumes. Definitions are systematically listed, and not carelessly scattered about. Proofs are given of all theorems except the simplest. Clear figures and illustrations are liberally employed. Exercises are select and abundant, but strong hints and even outlines of solution are given with each. Annexed to each chapter is a complete but concise summary.

A. G. FADELL

## NEW BOOKS RECEIVED

*Biometric Analysis, An Introduction*. By A. E. Treloar. Minneapolis, Minn., Burgess Publishing Company, 1951. v+251 pp.

*The New Physics*. By Sir C. V. Raman. New York, The Philosophical Library, 1951. 144 pp. $3.75.

*Secondary Mathematics, A Functional Approach for Teachers*. By H. F. Fehr. Boston, D. C. Heath and Co., 1951. xi+431 pp. $4.25.

*Differential Calculus*. By B. S. Ray. Calcutta, Das Gupta and Company, Ltd., 1950. ii+245 pp. Rs. 6/8.

*Dimensional Analysis and Theory of Models*. By H. L. Langhaar. New York, 1951. xi+166 pp. $4.00.

*Ontwakende Wetenschap*. By B. L. Van der Waerden. Groningen, P. Noordhoff N. V., 1950. 332 pp. $3.58.

*Infinite Matrices and Sequence Spaces*. By R. G. Cooke. London, MacMillan and Company, 1950, 13+347 pp. $6.50.

*Geometrie non Euclidienne, Par La Mathode Elementaire*. By Gustave Verriest. Paris, Gauthier-Villars, 1951. 192 pp. $3.12.

*Cours de Cinematique, Tome III, Geometrie et Cinematique Cayleyennes*. By Rene Garner. Paris, Gauthier-Villars, 1951. 376 pp. $8.89.

*Exercices de Mecanique*, Tome I, Fascicule II. By H. Beghin and Julia G. Beghin. Paris, Gauthier-Villars, 1951. 240 pp.

*Problemes Concrets D'Analyse Fonctionnelle*. By Paul Levy. Paris, Gauthier-Villars, 1951. 484 pp. $11.75.

*Intermediate Algebra*. By P. K. Rees and F. W. Sparks. New York, McGraw-Hill, 1951. viii+329 pp. $3.25.

*Hohere Algebra*, 3rd Ed. By Helmut Hasse. I Lineare Gleichungen. 152 pp. Hohere Algebra II, Gleichungen Hoheren Grades. 158 pp. Berlin, Walter de Gruyter, 1951. No price given.

*The Topology of Fibre Bundles* (No. 14, Princeton Mathematical Series). By Norman Steenrod, New Jersey, Princeton University Press, 1951. viii+224 pp. $5.00.

*Business Statistics*. By J. R. Riggleman and I. N. Frisbee. New York, McGraw-Hill Book Company, 1951. xix+818 pp. $5.50.

*An Introduction to Statistical Analysis*. By W. J. Dixon and F. J. Massey. New York, McGraw-Hill, 1951. x+370 pp. $4.50.

*National Income Behavior*. By T. C. Schelling. New York, McGraw-Hill, 1951. x+291 pp. $4.50.

*Trigonometry with Tables*. By C. T. Holmes. New York, McGraw-Hill, 1951. ix+246 pp. $3.00.

*Essentials of College Algebra*. By J. B. Rosenbach and E. A. Whitman, Boston, Ginn and Company, 1951. x+322 pp+xxx pp. $3.00.

*Johannes Kepler—Life and Letters*. By Carola Baumgardt. New York, The Philosophical Library, 1951. 209 pp. $3.75.

## CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

*Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.*

### Kappa Mu Epsilon, Eighth Biennial Convention

The Eighth Biennial Convention of *Kappa Mu Epsilon* was held in Springfield, Missouri on April 26–28, 1951 with *Missouri Alpha* chapter at Southwestern Missouri State College as host. A total of 188 delegates and members, representing thirty chapters and seventeen states, were registered. The president, Dr. H. Van Engen, presided at all sessions. His address as retiring president was called *You, Your Education and Kappa Mu Epsilon.*

The student papers presented to the convention were:

*The configurations* ($9_3$), by Hans Stetter of *Colorado Alpha*, at Colorado A. and M. College, Fort Collins

*Some properties of the Simpson line*, by Anne Robben, *Kansas Gamma* at Mount St. Scholastica College, Atchison

*The gamma function*, by Keith Stumpff, *Missouri Beta* at Central Missouri State College, Warrensburg

*The odd perfect number*, by James Barry, *Michigan Gamma* at Wayne University, Detroit

*The squirrel cage slide rule*, by Robert Green of *Kansas Alpha* at State Teachers College, Pittsburg

*Angle trisection*, by Wanda Ponder of *Iowa Alpha* at Iowa State Teachers College, Cedar Falls.

Dr. Leonard Blumenthal of the University of Missouri delivered the banquet address on *Relativity in mathematics.*

The officers elected for the coming biennium are: President, C. B. Tucker, State Teachers College, Emporia, Kansas; Vice-President, C. C. Richtmeyer, Central Michigan College of Education, Mount Pleasant, Michigan; Secretary, E. Marie Hove, Hofstra College, Hempstead, New York; Treasurer, L. F. Ollmann, Hofstra College, Hempstead, New York; Historian, Laura Greene, Washburn Municipal University, Topeka, Kansas.

Dr. H. D. Larsen, Albion College, Albion, Michigan was reappointed Editor of the *Pentagon*, the official publication of *Kappa Mu Epsilon.*

The delegates to the convention voted to incorporate *Kappa Mu Epsilon*, Honorary Mathematics Society under the laws of the State of New York.

The Ninth Biennial Convention will be held at St. Mary's Lake in Michigan on April 17–18, 1953 with the Michigan Chapters as hosts.

### CLUB REPORTS

#### Mathematics Club, Kansas State College

During the year 1949–50, the *Mathematics Club* of Kansas State College had

a series of seven talks by people in the Department of Mathematics. The speakers and their topics were as follows:

*Calendar problems*, by Lois Carper

*Fourier analysis by a method due to Stokes*, by Prof. W. C. Janes

*A method of testing hypotheses*, by Robert Cell

*Series*, by Dr. S. T. Parker

*Bessel functions*, by Lansford Trapp

*The theory of groups and their representation of matrices*, by Violet Hachmeister

*Pascal's triangle*, by John Neff.

The officers for the year were: President, Wayne Cowell; Vice-President, Lansford Trapp; Secretary-Treasurer, Raymond Rinker.

**Undergraduate Mathematics Society, Columbia College**

The *Undergraduate Mathematics Society* of Columbia College met weekly during 1950–51 for the presentation of papers, the aim of which was to conclude, usually, with some significant result. The following lectures were given:

*The transcendence of e*, by M. Epstein

*Peano's axioms*, by C. Langley

*Fixed point theorems*, by S. Stein

*The four-color problem*, by I. Mann

*Group theory*, by R. Feldman

*The calculus of finite differences*, by C. Kaufman

*Embedding theorems*, by A. Fass

*Transfinite cardinals*, by Dr. M. Auslander

*The concept of homotopy*, by Prof. S. Eilenberg.

The officers for the year were: President, E. Auerbach; Vice-President, C. Kaufman.

**Pi Mu Epsilon, Louisiana State University**

The *Louisiana Alpha* chapter of *Pi Mu Epsilon* heard the following papers for 1950–51:

*The number system*, by Prof. F. A. Rickey

*Plane continua*, by Prof. N. E. Rutt

*Development of quantum mechanics*, by Prof. V. E. Parker

*Some applications of mathematical concepts to chemical problems*, by Prof. H. B. Williams

*Groups in crystal structure and theory of equations*, by Prof. Eugene Schenkman

*Another way of doing it*, by Prof. P. K. Rees

*Calculus of variations*, by Prof. B. B. Townsend

*The exterior differential calculus of Cartan*, by Prof. Eugenio Calabi.

Twenty-one members were initiated at the annual Banquet and Initiation.

Prof. C. C. MacDuffee, Director-General of *Pi Mu Epsilon*, gave the banquet address as part of the *Pi Mu Epsilon Lectures* for the year. The second address on *Curves in Minkowski space* was given the next day.

The following awards were presented at the banquet: Herbert W. Kelley received the Freshman Award, based on an Honors Examination; Horace C. Hearne and Roger W. Richardson, Jr. tied in the Senior Award, based on the amount of work taken in mathematics and the quality of work done.

Several new volumes were added to the Pi Mu Epsilon shelf during the year.

The officers serving for 1950–51 were: Director, James M. Turner; Vice-Director, Horace C. Hearne; Secretary, Delilah Stokes; Treasurer, Roger W. Richardson; Corresponding Secretary and Faculty Advisor, Prof. Houston T. Karnes.

**Mathematics Club, Syracuse University**

The plans for the Syracuse University *Mathematics Club* for 1950–51 include:

The publishing of a Club Paper to appear tri-semesterly. The paper will include a Problem Department.

Other plans call for the issuance of membership cards to the members.

The officers elected for 1950–51 are: President, Sherman Babcock; Vice-President, Harry Kagan; Secretary, Vera Rita; Treasurer, Frank Manley.

## NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.*

### ANNUAL CONVENTION OF THE CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS

The annual Convention of the Central Association of Science and Mathematics Teachers will be held in Cleveland, Ohio, on November 22–24, 1951. The Hollenden Hotel has been selected for headquarters.

Interesting general meetings are scheduled for Friday morning and afternoon, November 23, and Saturday morning, November 24. Among the featured general speakers are the following: Dr. Keith Glennan, President of Case Institute of Technology and member of the Atomic Energy Commission; Dr. Elvin C. Stakman, Chief of the Division of Plant Pathology and Botany, University of Minnesota; and Dr. David Dietz, Science Editor, Scripps Howard Newspapers.

Dr. Dietz will speak also at the banquet Friday evening.

Section meetings in Elementary Mathematics, Elementary Science, General Science, Mathematics, Biology, Chemistry and Physics will be held Friday.

Group meetings in areas of Elementary Schools, Junior High Schools, Senior High Schools, and Conservation will be held on Saturday.

### A NEW SCIENTIFIC JOURNAL

A new scientific journal, *Current Research*, will be published to furnish a medium for the prompt publication of brief papers recording the progress of science.

The journal is designed to serve the scientist in ways not done by any existing publication. Its unique features are: (1) Articles limited to 1,000 words; (2) Printed on one side of a single page; (3) Summary title printed along right-hand margin; (4) Inner edge of page perforated to facilitate easy removal and filing; (5) Authors assessed $10.00 for each article, enabling the journal to expand as necessary for prompt publication (15–30 days); (6) No editorial board; (7) Scope: the entire field of science; (8) International in character.

The first issue will be published on October 5, 1951. The subscription rate is $5.00. The editor is Ware Cattell, formerly editor of *The Scientific Monthly* and assistant editor of *Science*.

### FELLOWSHIPS OF THE AMERICAN ASSOCIATION OF UNIVERSITY WOMEN

Twenty-five fellowships are offered by the American Association of University Women to American women for advanced study or research during the academic year 1952–53.

In general, the $1,500 fellowships are awarded to young women who have completed two years of residence work for the Ph.D. degree or who have already received the degree; the $2,000–$2,200 awards to more advanced students or those who may need to study abroad; the $3,000 awards to more mature scholars who need a year of uninterrupted work for writing and research. Unless otherwise specified, the fellowships are unrestricted as to subject and place of study.

Applications and supporting materials must reach the office in Washington by December 15, 1951. For detailed information concerning these fellowships and instructions for applying, address the Secretary, Committee on Fellowship Awards, American Association of University Women, 1634 Eye Street, N. W., Washington 6, D. C.

### PERSONAL ITEMS

Professor E. M. Beesley, University of Nevada, was the representative of the Association at the Inauguration of President M. A. Love of the University of Nevada.

Professor Leonard Blumenthal of the University of Missouri represented the Association at the Inauguration of Dr. R. L. Woodward as President of

Central College on June 2, 1951.

Professor L. J. Green of Case Institute of Technology was appointed to represent the Association at the celebration of the One Hundred Twenty-Fifth Anniversary of Western Reserve University on June 11, 1951.

Professor F. E. Johnston, George Washington University, represented the Association at the thirty-fourth Annual Meeting of the American Council of Education which was held in Washington, D. C. on May 4–5, 1951.

Professor F. J. Murray of Columbia University was the delegate of the Association to the Academic Convocation at Manhattan College on April 30, 1951; this Convocation commemorated the Tercentenary of the birth of Saint John Baptist de la Salle.

Professor W. R. Ransom of Tufts College represented the Association at the Inauguration of President H. C. Case of Boston University on June 3, 1951.

Dr. Christine Williams Ayoub of Cornell University has been awarded the Sigma Delta Epsilon Post-Doctoral Fellowship for 1951–52.

The officers of the Mathematics Division of the American Society for Engineering Education for 1951–52 are: Professor H. M. Gehman, Chairman and Professor J. H. Zant, Secretary.

Professor H. P. Robertson, California Institute of Technology, and Professor G. T. Whyburn of the University of Virginia have been elected to membership in the National Academy of Sciences.

Cooper Union announces the following promotions in the School of Engineering: Assistant Professor J. N. Eastham to an associate professorship and Mr. C. H. Lehmann to an assistant professorship.

At Fresno State College: Professor F. R. Morris has been appointed Head of the Division of Physical Sciences; Associate Professor Alice K. Bell has been promoted to a professorship.

Kansas State College announces: Professor W. T. Stratton has retired after forty-one years service with the title of Professor Emeritus; Associate Professor Emma Hyde has retired after thirty-one years service with the title of Associate Professor Emeritus; Associate Professors P. M. Young and S. T. Parker have been promoted to professorships; Associate Professor Edison Greer has resigned to accept a position with Beech Aircraft, Wichita, Kansas; Mrs. Marilynn S. Dueker has resigned.

Kent State University announces the following promotions: Associate Professor Emma Olson to a professorship; Assistant Professor Paul Evans to an associate professorship; Miss Emalou Brumfield to an assistant professorship.

Massachusetts Institute of Technology reports the following: Professor W. T. Martin, chairman of the Department of Mathematics, will be at the Institute for Advanced Study during 1951–52; Assistant Professor G. W. Whitehead has been promoted to an associate professorship; Mr. A. H. Copeland, Jr., Mr. Felix Haas, Mr. E. B. Leach, Mr. P. P. Radkowski, and Mr. J. M. Stark have been promoted to instructorships.

University of Washington announces the following: Professor Z. W. Birn-

baum has been granted a leave of absence to accept a visiting professorship at Stanford University for 1951–52; Professor Edwin Hewitt has been granted a leave of absence for the academic year 1951–52 to accept an appointment as visiting lecturer at the University of Uppsala; Dr. D. B. Dekker has been promoted to an assistant professorship; Dr. M. G. Arsove, who has been studying at Grenoble on a Fulbright grant, has been appointed to an instructorship.

At Western Washington College of Education: Associate Professor S. A. Johnston, chairman of the Department of Mathematics, has been promoted to a professorship; during the summer Professor Johnston was a member of the staff of California Institute of Technology; Instructor H. M. Gelder has been promoted to an assistant professorship.

Assistant Professor Henry Antosiewicz of Montana State College has been promoted to an associate professorship.

Mr. J. S. Arend, formerly a part-time instructor at the University of Colorado, is now in the United States Air Force.

Dr. R. C. Briant, who has been at the Institute for Cooperative Research of Johns Hopkins University, has accepted the position of Project Director, Oak Ridge National Laboratory, Tennessee.

Mrs. Helen E. Brown, formerly an instructor at the University of Tennessee, is teaching at Fassifern School for Girls, Henderson, North Carolina.

Assistant Professor R. L. Calvert of the University of Wyoming has received an appointment in the Research Division of Sandia Corporation, Albuquerque, New Mexico.

Miss Mary Campbell of Lamar College retired on June 1, 1951.

Graduate Assistant JoAnn M. Cumming of Marquette University has a position as mathematician at the Aberdeen Proving Ground, Maryland.

Instructor F. W. Donaldson of the University of Texas has accepted a position on the technical staff of the Los Alamos Scientific Laboratory, New Mexico.

Dr. Nat Edmonson of Fairchild Engine and Airplane Corporation, Oak Ridge, Tennessee, has been appointed to the position of Principal Mathematician at the Oak Ridge National Laboratories.

Assistant Professor W. L. Fields of the University of Louisville has been appointed to an associate professorship at Hampton Institute.

Assistant Professor J. R. Foote, Iowa State College, is now at Wright-Patterson Air Force Base, Dayton, Ohio.

Professor L. R. Ford, chairman of the Department of Mathematics of Illinois Institute of Technology, was honored at a special meeting of the Men's Mathematics Club of Chicago on May 18, 1951.

Mr. J. H. Fountain, formerly a graduate student at the University of Buffalo, is now a research chemist with the Upson Company, Lockport, New York.

Mr. E. A. Franz of Culver-Stockton College has been recalled to active duty in the United States Army Signal Corps.

Mr. W. H. From, previously a graduate student at the University of Virginia, has accepted a position as Research Engineer at Aircraft Armaments

Incorporated, Baltimore, Maryland.

Professor W. H. Garrett of Baker University has retired with the title of Professor Emeritus of Mathematics and Astronomy; Professor Garrett was one of the founders of the Kansas Association of Teachers of Mathematics.

Mr. G. H. Gleissner, Columbia University, has a position as mathematician at the United States Naval Proving Ground, Dahlgren, Virginia.

Mr. R. B. Grekila, formerly a graduate assistant at the University of Maryland, has accepted a position as Assistant Chief Chemist at Ball Brothers Company, Incorporated, Muncie, Indiana.

Professor S. G. Hacker, State College of Washington, has been on leave of absence and has been serving as a mathematical consultant with Operations Analysis, Strategic Air Command, United States Air Force, Omaha, Nebraska.

Mr. P. C. Hanzel, formerly laboratory assistant at the University of California, has accepted a position as a research assistant in the Cryogenic Laboratory, Ohio State University.

Mr. Julius Honig of Raytheon Electronic Corporation, Waltham, Massachusetts, has a position as mathematician at the National Bureau of Standards, Washington, D. C.

Mrs. Verba W. Iturralde, previously an instructor at the College of William and Mary, is teaching in the El Paso Public Schools, Texas.

Dr. Hyman Kaufman of the Geophysical Department, Continental Oil Company, Ponca City, Oklahoma, has accepted a position in the Laboratory for Electronics, Incorporated, Boston, Massachusetts.

Mr. J. B. Kelly, University of Wisconsin, has been appointed a member of the Institute for Advanced Study.

Mr. R. S. Kingsbury, formerly a staff member of Operations Evaluation Group, Navy Department, Washington, D. C., is now a member of Operations Research Staff of Arthur D. Little, Incorporated, Cambridge, Massachusetts.

Visiting Assistant Professor Jacob Korevaar of Purdue University has been appointed to a professorship at the Institute of Technology, Delft, Netherlands.

Mrs. Mary M. Kruse of the Sandia Corporation, Albuquerque, New Mexico, has a position as mathematician at the Institute for Numerical Analysis, Los Angeles, California.

Assistant Professor Werner Leutert has been granted a leave of absence from the University of Maryland and has accepted a position as mathematician at the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

Mr. H. C. Mayer, Jr. of the University of Utah has a position as mathematician with the General Electric Corporation, Richland, Washington.

Dr. J. C. C. McKinsey, formerly a mathematician at the Rand Corporation, Santa Monica, California, has been appointed Visiting Professor of Philosophy at Stanford University.

Dr. C. E. Mullan of the Duquesne Light Company, Pittsburgh, Pennsylvania, and of the Carnegie Institute of Technology has been appointed Servomechanisms Engineer, Bell Aircraft Corporation, Buffalo, New York.

Professor P. R. Rider has a leave of absence from Washington University for the academic year 1951–52 to act as Mathematical and Statistical Adviser to the Chief of the Office of Air Research, Air Development Force, Air Research and Development Command, Wright-Patterson Air Force Base, Dayton, Ohio.

Assistant Professor Saul Rosen of Drexel Institute of Technology has joined the Research Division of the Burroughs Adding Machine Company, Philadelphia, Pennsylvania.

Assistant Professor W. C. Sangren, University of Miami, has a position as senior mathematician at the Oak Ridge National Laboratory, Tennessee.

Mr. R. J. Smurthwaite, formerly a graduate assistant at Lehigh University, has a position at Bell Aircraft Corporation, Buffalo, New York.

Dr. E. V. Somers of Westinghouse Electric Corporation, Pittsburgh, Pennsylvania, is now Senior Research Technologist, Magnolia Petroleum Company, Dallas, Texas.

Instructor R. D. Stalley of the University of Arizona has been appointed to a graduate assistantship at the University of Oregon.

Mr. W. J. Strauss has accepted a position as junior mathematician at the Air Weapons Research Center of the University of Chicago.

Dr. G. W. Tyler of the Navy Electronic Laboratory, San Diego, California, has a position as operations analyst, United States Air Force, Washington, D. C.

Dr. J. L. Ullman is on leave of absence from the University of Michigan and will be located at Stanford, California, during the year 1951–52.

Professor J. W. T. Youngs is on leave of absence from Indiana University and has a position as research consultant for the Atomic Energy Commission at Sandia Corporation, Albuquerque, New Mexico.

Professor G. A. Bliss, who was Martin A. Ryerson Distinguished Service Professor of Mathematics Emeritus at the University of Chicago, died on May 8, 1951. He was a charter member of the Association.

Reverend Hilary Doerfler of St. Gregory's College died on March 18, 1950.

Professor C. H. Gingrich, chairman of the Department of Mathematics, Carleton College, died on June 17, 1951. He had been a member of the Association for thirty-three years.

Professor T. W. Jackson, head of the Department of Mathematics, Jamestown College, North Dakota, died on June 10, 1950. He was a charter member of the Association.

Dr. R. B. Robbins of Charlottesville, Virginia, died on February 11, 1951.

Professor Emeritus W. D. A. Westfall of the University of Missouri died on April 28, 1951.

Professor Emeritus A. H. Wheeler of Clark University died on December 19, 1950. He was a charter member of the Association.

## THE MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

### NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following ninety-two persons have been elected to membership by the Board of Governors on applications duly certified.

R. C. AILARA, B.S.E.E. (Illinois Tech.) Grad. Student, Illinois Institute of Technology, Chicago, Ill.

W. H. AMMANN, M.S. (St. Louis) Grad. Student, St. Louis University, Mo.

R. F. ATTRIDGE, Student, McMaster University, Hamilton, Ont.

FREDERICK BAGEMIHL, M.A. (Columbia) Asst. Professor, University of Rochester, N. Y.

J. J. BAILEY, M.A. (Michigan) Acting Head of Mathematics Department, Shattuck School, Faribault, Minn.

R. E. BAKER, M.A. (Texas) Asso. Professor, East Texas State Teachers College, Commerce, Tex.

R. S. BARTON, M.S. (Iowa) Instr., New Mexico School of Mines, Socorro, N. M.

T. F. BOBLAK, B.S. (Illinois Tech.) Grad. Student, Illinois Institute of Technology, Chicago, Ill.

J. V. BREGLIO, Student, George Pepperdine College, Los Angeles, Calif.

J. D. BRERETON, M.A. (Columbia) Vice-Principal, New Paltz High School, N. Y.

CHARLOTTE BROWN, M.S. (N.Y.U.) Part-time Instr., Brooklyn College, N. Y.

H. H. BROWN, A.B. (Penna. State) Research Engineer, Franklin Institute Laboratories, Philadelphia, Pa.

MARJORIE L. BROWNE, Ph.D. (Michigan) Instr., North Carolina College, Durham, N.C.

B. H. CHOVITZ, M.A. (Harvard) Mathematician, Army Map Service, Washington, D. C.

R. A. CHRISTENSEN, Student, Willamette University, Salem, Ore.

MRS. WILLIE E. CLARK, M.A. (Atlanta) Head of Mathematics Department, Agricultural Mechanical and Normal College, Pine Bluff, Ark.

R. C. CLELLAND, A.M. (Columbia) Instr., Hamilton College, Clinton, N. Y.

P. J. COCUZZA, Student, City College of the City of New York, N. Y.

M. L. COFFMAN, M.A. (Oklahoma) Instr., East Texas State Teachers College, Commerce, Tex.

ECKFORD COHEN, Ph.D. (Duke) Asst. Professor, Syracuse University, N. Y.

C. H. COOK, M.S. (Iowa) Asst. Professor, Western State College, Gunnison, Colo.

J. W. CORNWELL, 3RD, B.A. (Dartmouth) Senior Engineer, The Teleregister Corporation, New York, N. Y.

R. B. CROUCH, M.S. (Illinois) Instr., New Mexico College of Agricultural and Mechanic Arts, State College, N. M.

MRS. ELIZABETH H. CUTHILL, M.S. (Brown) Instr., Purdue University, Lafayette, Ind.

G. H. DUBAY, M.A. (Loyola, Chicago) Professor, University of St. Thomas, Houston, Tex.

D. L. ELLIOTT, Student, Pomona College, Claremont, Calif.

R. L. ELY, M.S. (C.I.T.) Research Engineer, Pittsburgh-Des Moines Company, Pittsburgh, Pa.

H. M. EMICH, Student, University of Rochester, N. Y.

D. O. ETTER, B.A. (Texas Christian) Grad. Student, Texas Christian University, Fort Worth, Tex.

BEN FITZPATRICK, JR., Student, Alabama Polytechnic Institute, Auburn, Ala.

D. D. FREDERICK, Student, George Pepperdine College, Los Angeles, Calif.

W. H. FUCHS, Ph.D. (Cambridge, England) Asso. Professor, Cornell University, Ithaca, N. Y.

F. W. GIBSON, Student, Los Angeles State College, Calif.

R. C. GILBERT, A.B. (Harvard) Engineering Technician, Lockheed Aircraft Corporation, Burbank, Calif.

F. J. GINIVAN, M.S. (Notre Dame) Teaching Fellow, University of Notre Dame, Ind.

M. L. GOLDWATER, M.A. (U.C.L.A.) Los Angeles, Calif.

E. P. GRANEY, B.S. (Rockhurst) Grad. Student, University of Notre Dame, Ind.

J. H. GRIESMER, Student, University of Notre Dame, Ind.

G. R. HAGEN, Student, University of Kentucky, Lexington, Ky.

CARL HAMMER, Ph.D. (Munich) Chairman, Division of Technical Education, Walter Hervey Junior College, New York, N. Y.

F. R. HARDING, Student, Marquette University, Milwaukee, Wis.

H. G. HARP, M.S. (Chicago) Asso. Professor, Ohio Northern University, Ada, Ohio

CLINTON HELTON, Student, Eastern Kentucky State College, Richmond, Ky.

J. E. HICKS, M.S. (DePaul) Teacher, Chicago Board of Education, Ill.

EVA JAGOE, B.A. (Alberta) Teacher, Calgary School Board, Alta.

W. J. JAMESON, Student, Montana State University, Missoula, Mont.

PHILIP JONAS, B.S. (Brooklyn C.) Instr., RCA Institutes, New York, N. Y.

F. A. KROS, M.S. (New Mexico A & M) Instr., University of Colorado, Boulder, Colo.

M. T. LANKALIS, B.S.E. (S.T.C., Bridgewater, Mass.) Instr., New Hampshire Technical Institute, Manchester, N. H.

E. H. LEE, Ph.D. (Stanford) Asso. Professor, Brown University, Providence, R. I.

G. R. LEHNER, Student, Loyola University, Chicago, Ill.

B. L. LERCHER, Student, University of Wisconsin, Madison, Wis.

MRS. NORMA L. LINDEMANN, A.M. (Indiana) Research Assistant, University of Notre Dame, Ind.

D. B. LLOYD, Ph.D. (Catholic) Instr., Wilson Teachers' College, Washington, D. C.

R. G. MCDERMOT, B.S. (Pittsburgh) Instr., Western Pennsylvania Institute of Technology, Pittsburgh, Pa.

J. A. MEIER, Student, Franklin and Marshall College, Lancaster, Pa.

R. B. MERKEL, A.B. (Sacramento S. C.) Instr., Sacramento Unified School District, Calif.

GEORGE MILLMAN, M.A. (Columbia) Mathematician, Army Map Service, Corps of Engineers, Washington, D. C.

J. T. MONTGOMERY, B.S. (Notre Dame) Teaching Fellow, University of Notre Dame, Ind.

T. D. OXLEY, Jr., Student, Texas Christian University, Fort Worth, Tex.

W. O. PORTMANN, Student, Kent State University, Ohio

CONRAD RENNEMANN, JR., B.S. (Nebraska) Grad. Assistant, University of Nebraska, Lincoln, Neb.

J. B. RYAN, B.S. (Seattle) Treasurer, Rice High School, New York, N. Y.

RAFAEL SANCHEZ-DIAZ, Ph.D. (California) Professor, New Mexico School of Mines, Socorro, N. M.

B. E. SEALANDER, B.S. (Seton Hall) Senior Research Engineer, Republic Aviation, Farmingdale, L. I., N. Y.

R. Q. SEALE, Ph.D. (Stanford) Professor, Southern Methodist University, Dallas, Tex.

SISTER MARY OLIVE NORMAN, M.S. (Marquette) Head of Mathematics Department, Cardinal Stritch College, Milwaukee, Wis.

SISTER MIRIAM PATRICK, B.S. (St. Mary's Indiana) Grad. Assistant, St. Mary's College, Notre Dame, Ind.

A. H. SMITH, Student, George Pepperdine College, Los Angeles, Calif.

J. D. SPEAS, Student, Butler University, Indianapolis, Ind.

H. E. SPEECE, M.A. (Texas Christian) Instr., North Carolina State College, Raleigh, N. C.

F. M. STEWART, Ph.D. (Harvard) Asst. Professor, Brown University, Providence, R. I.

S. SUBRAMU, M.Sc. (Mysore) Lecturer, Technical Institute, Tata Iron and Steel Company, Jamshedpur, Bihar, India

R. R. SUTTER, Student, Illinois Institute of Technology, Chicago, Ill.

W. W. TAYLOR, M.S. (East Texas S. T. C.) Instr., East Texas State Teachers College, Commerce, Tex.

G. C. THOMPSON, B.S. in E.E. (Union C., New York) Asso. Actuary, Security Mutual Life Insurance Company, Binghamton, N. Y.

F. J. WAGNER, M.S. (St. Louis) Grad. Fellow, University of Notre Dame, Ind.

J. G. WALL, M.A. (North Carolina) Grad.

Assistant, University of Georgia, Athens, Ga.

G. M. WEBSTER, M.A. (Oklahoma) Research Physicist, Carter Oil Company, Tulsa, Okla.

A. M. WENNER, Student, Gustavus Adolphus College, St. Peter, Minn.

R. A. WERDELIN, Student, Trinity College, Hartford, Conn.

H. W. WESSELY, Student, Illinois Institute of Technology, Chicago, Ill.

C. S. WEST, JR., B.S. (Southern Methodist) Assistant to the Actuary, Guardian International Life Insurance Company, Dallas, Tex.

H. E. WHEELER, Student, Northwestern University, Evanston, Ill.

R. S. WOLLAN, B.A. (Luther C., Iowa) Grad. Student, University of Georgia, Athens, Ga.

P. G. WOOD, B.Sc. (London) Mathematics Master, Eastleigh County High School, Hampshire, England

DALE WOODS, M.S. (Oklahoma A & M) Instr., North Dakota Agricultural College, Fargo, N. D.

T. M. WRIGHT, M.A. (Michigan) Asso. Professor, Berea College, Ky.

MELVIN YEDLIN, B.S. (Missouri) Automotive Design Engineer, Detroit Arsenal, Centerline, Mich.

R. L. YOUNG, B.S. (Davis and Elkins) Grad. Assistant, University of Denver, Colo.

E. J. ZIRKEL, Student, St. John's University, Brooklyn, N. Y.

M. N. ZUCKER, Student, New York University, N. Y.

## NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1951 by a mail vote of the membership of the Association in the Sections indicated:

| | |
|---|---|
| Allegheny Mountain | J. B. Rosenbach, Carnegie Institute of Technology |
| Indiana | J. C. Polley, Wabash College |
| Kentucky | Aughtum S. Howard, Kentucky Wesleyan College |
| Metropolitan New York | T. F. Cope, Queens College |
| Nebraska | J. M. Earl, University of Omaha |
| Northern California | E. B. Roessler, University of California at Davis |
| Oklahoma | J. C. Brixey, University of Oklahoma |
| Rocky Mountain | C. F. Barr, University of Wyoming |
| Wisconsin | H. P. Pettit, Marquette University |

Of the forty-one members of the Board of Governors, twenty-six Governors are elected as representatives of the Sections of the Association. In this way the Association attempts to make participation in its activities as widely spread as possible among its membership.

Each year the members in each of the Sections in which an election is being held show great interest in the conduct of the election. In the 1951 election, in six of the nine Sections more than half of those eligible to vote case their ballots in time to be counted. The highest percentage of votes cast was 74% in the Kentucky Section.

H. M. GEHMAN, *Secretary-Treasurer*

## JOINT MEETING OF THE ASSOCIATION WITH A.S.E.E.

A meeting of the Mathematical Association of America was held at Michigan State College, East Lansing, Michigan, on Monday and Tuesday, June 25–26,

1951, in conjunction with the annual meeting of the American Society for Engineering Education. All sessions of the Association were joint sessions with the Mathematics Division of the A.S.E.E.

About one hundred persons attended the sessions of the Association including the following sixty members of the Association:

R. P. Agnew, J. W. Baldwin, J. M. Barbour, W. D. Baten, J. H. Bell, S. K. Berberian, J. B. Brandeberry, D. M. Brown, R. E. Carr, R. V. Churchill, J. B. Coleman, J. W. Coy, P. H. Daus, J. E. Dotterer, W. E. Felling, H. E. Fettis, K. W. Folley, Clarence Ford, J. S. Frame, R. W. Gardner, H. M. Gehman, V. G. Grove, G. E. Hay, Fritz Herzog, J. F. Heyda, D. L. Holl, L. P. Hutchison, L. S. Johnston, P. S. Jones, H. K. Justice, L. M. Kelly, W. C. Krathwohl, John Kronsbein, A. C. Ladner, Leo Lapidus, Z. L. Loflin, Florence Long, G. H. MacCullough, Saunders MacLane, L. E. Mehlenbacher, D. M. Mesner, F. H. Miller, Harriet F. Montague, Mabel D. Montgomery, B. C. Moore, A. B. Neale, C. V. Newsom, E. A. Nordhaus, Mary H. Payne, L. C. Plant, J. E. Powell, W. D. Rice, Joseph Spear, E. J. Specht, H. E. Stelson, B. M. Stewart, P. C. Sweetland, H. W. Syer, C. P. Wells, E. A. Whitman.

A banquet was held on Monday at 6:00 P.M. in Parlor A of the Union Building with President Saunders MacLane presiding. Professor W. C. Krathwohl, Chairman of the Mathematics Division, A.S.E.E., presided at the First Session on Tuesday afternoon in Room 120, Physics-Mathematics Building. The Second Session was held in Room 118, Physics-Mathematics Building on Tuesday evening with Professor G. E. Hay presiding. The Program Committee for the meeting consisted of G. E. Hay, Chairman D. F. Gunder, and F. H. Miller.

## BANQUET

"Some Problems in Connection with the Teaching of Elementary and Secondary Mathematics," by Dr. C. V. Newsom, Associate Commissioner for Higher and Professional Education, State of New York.

## FIRST SESSION

"Selected Formulas for Numerical Approximation," by Professor J. S. Frame, Michigan State College.

"Some Analogies in Elasticity Problems," by Professor D. L. Holl, Iowa State College.

"Integral Transforms and Differential Equations," by Professor R. V. Churchill, University of Michigan.

## SECOND SESSION

"Views on Differential Equations," by Professor R. P. Agnew, Cornell University.

"Curves of Pursuit," by Professor K. W. Folley, Wayne University.

"Periodic Constructions," by Professor B. M. Stewart, Michigan State College.

## ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of J. H. Bell, Chairman, H. M. Gehman, H. E. Stelson, B. M. Stewart, C. P. Wells, and J. H. Zant.

Registration headquarters were located in Shaw Hall of Michigan State College. Rooms in Shaw and Phillips dormitories were available for members of the Association and their families. Meals were served in the dormitory cafeterias.

A tea was given by the Mathematics Department of Michigan State College in the Conference Room of the Physics-Mathematics Building at the conclusion of the Tuesday afternoon session.

Other entertainment was provided under the auspices of the A.S.E.E. Visits to local automobile plants were arranged for each afternoon.

Sessions of the A.S.E.E. began on Monday and continued through Friday. In addition to the sessions mentioned above, the Mathematics Division of the A.S.E.E. met jointly with the Educational Methods Division on Wednesday afternoon.

At a brief business meeting of the Association, held on Tuesday evening, a motion was presented by Professor P. H. Daus expressing the appreciation of the members of the Association and of the Division for the hospitality of the authorities of Michigan State College and especially the local members of the Committee on Arrangements whose efforts had made possible this successful meeting.

H. M. GEHMAN, *Secretary-Treasurer*

## MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The thirty-first regular meeting of the Southern California Section of the Mathematical Association of America was held at Whittier College, Whittier, California, on Saturday, March 10, 1951. Professor Herbert Busemann, Chairman of the Section, presided.

The attendance was eighty-four, including the following fifty-eight members of the Association: L. J. Adams, J. T. Ahlin, O. W. Albert, Florence R. Anderson, L. A. Aroian, Winifred A. Asprey, Lulu Bechtolsheim, May M. Beenken, C. M. Bell, Jr., Jonas Beraru, L. T. Black, Herbert Busemann, Frances L. Campbell, L. M. Coffin, W. V. Gamzon, W. H. Glenn, Jr., W. T. Guy, Jr., C. J. A. Halberg, H. J. Hamilton, V. C. Harris, A. F. Herbst, R. B. Herrera, M. R. Hestenes, P. G. Hoel, R. E. Horton, D. H. Hyers, C. G. Jaeger, P. B. Johnson, P. J. Kelly, Cornelius Lanczos, L. C. Lay, Margaret B. Lehman, G. F. McEwen, F. R. Morris, P. M. Niersbach, L. J. Paige, D. J. Peterson, W. T. Puckett, C. A. Pursel, H. R. Pyle, E. C. Rex, J. M. Robb, R. W. Shephard, Samuel Skolnik, J. C. Smith, Jr., R. H. Sorgenfrey, D. V. Steed, Robert Steinberg, E. B. Strutton, J. D. Swift, T. E. Sydnor, A. E. Taylor, S. E. Urner, F. A. Valentine, Morgan Ward, P. A. White, A. L. Whiteman, B. R. Wicker.

At the business meeting the following officers were elected for the academic

year 1951–52: Chairman, W. H. Glenn, Jr., Pasadena City Schools; Vice-Chairman, C. W. Trigg, Los Angeles City College; Program Committee, D. H. Hyers (Chairman) University of Southern California, P. B. Johnson, Occidental College, E. C. Rex, George Pepperdine College. The next meeting was scheduled for March 8, 1952, at Occidental College, Los Angeles, California.

The following program was presented:

1. *Unified freshman mathematics*, by Professor S. E. Urner, Los Angeles State College, Professor F. A. Valentine, University of California at Los Angeles, Professor H. R. Pyle, Whittier College.

2. *The nominalistic interpretation of mathematics*, by Professor L. A. Henkin, University of Southern California, introduced by Professor P. A. White.

The problem considered is: To provide a systematic method of reinterpreting the language of mathematics in such a way that no reference is made to abstract objects such as sets, functions, numbers, etc. It is required that logical terms such as "not," "if . . . then," "for every $x$," etc., retain their basic meanings, and that sentences which can be proved by ordinary methods of mathematical proof shall express true propositions under the new interpretations of the symbols. Using methods of mathematical logic, it is shown that there are a multiplicity of solutions to this problem.

3. *Mapping of the integers onto themselves preserving the partial ordering of division*, by Professor Morgan Ward, California Institute of Technology.

Mappings $\phi=\phi_n$ of the integers into themselves such that $\phi_n$ divides $\phi_m$ whenever $n$ divides $m$ are of frequent occurrence in the theory of numbers. A general theory of such mappings is here developed with emphasis on their lattice properties with respect to the partial ordering by division. Of particular interest are those mappings $K$ with the two additional properties (i) $K_0=0$, $K_1=1$, $K_n\neq 0$, $n\neq 0$; (ii) For every integer $m$, the set of integers $x$ such that $K_x\equiv 0 \pmod m$ is an ideal. (The Lucas function $(\alpha^n-\beta^n)/(\alpha-\beta)$, $\alpha+\beta$, $\alpha\beta$ coprime integers, is a $K$-mapping.) Such mappings have a number of interesting arithmetical properties. For example a $\phi$-mapping with $\phi_n\neq 0$, $n\neq 0$, $\phi_0=0$, $\phi_1=1$ is a $K$-mapping if and only if $(\phi_n, \phi_m)=\phi((n, m))$. (Here the parentheses $(\ldots)$ indicate a greatest common divisor.)

4. *Experience of a Fulbright professor in England*, by Professor Elmer Tolsted, Pomona College, introduced by Professor C. G. Jaeger.

This speaker gave a description of the Fulbright Scholarship, and also discussed the general standard of living and teachers salaries in England, the B.A. requirements in mathematics at the University of London, and devices of the English to keep political influence out of their schools.

5. *Monte Carlo solution of linear equations*, by Dr. G. E. Forsythe, National Bureau of Standards, Los Angeles, California.

A method of solving certain classes of systems of linear equations (or inverting the coefficient matrices) by random sampling was devised by J. von Neumann and S. M. Ulam (unpublished). The method was expounded and extended by George E. Forsythe and Richard A. Leibler, *Mathematical Tables and Other Aids to Computation*, vol. 4 (1950), pp. 127–129, and vol. 5 (1951), p. 55. The present speaker explained the method and, with the assistance of Professor Paul G. Hoel, demonstrated it on a simple system with two unknowns.

6. *The Cobb-Douglas production function*, by Dr. R. W. Shephard, Rand Corporation.

The origin, economic meaning, and statistical analyses of the Cobb-Douglas production function are briefly reviewed. Following this, a mathematical development is given which furnishes a precise foundation for this well known empirical function and provides definitions for the aggregate quantities involved leading to the usual interpretations of the parameters. Thereby a rigorous foundation is provided for the function, enabling simplifications of the statistical analyses made heretofore. An exact correspondence between ratio of wage bill to total value added and the labor elasticity of output is found. As a by-product of the mathematical development, the cost function corresponding to the Cobb-Douglas production function is found to have equally simple expression.

W. T. PUCKETT, *Acting Secretary*

## MARCH MEETING OF THE SOUTHWESTERN SECTION

The eleventh annual meeting of the Southwestern Section of the Mathematical Association of America was held at the University of New Mexico, Albuquerque, New Mexico, on March 23 and 24, 1951.

Dr. A. W. Boldyreff, Chairman of the Section, presided. Fifty-five persons attended the sessions, including the following thirty-eight members of the Association: J. W. Beach, A. W. Boldyreff, C. E. Buell, J. H. Butchart, C. R. Clark, G. A. Culpepper, Angelo Di Bella, D. G. Duncan, R J. Flanagan, R. S. Fouch, F. C. Gentry, R. F. Graesser, P. C. Hammer, W. P. Heinzman, M. S. Hendrickson, R. C. Hildner, B. W. Jones, Max Kramer, G. N. Landes, F. O. Lane, Rachael A. La Roe, L. E. Mahuron, R. W. Malley, H. V. Mathany, J. J. Miller, W. W. Mitchell, Leo Moser, J. W. Reed, N. W. Riebe, B. D. Roberts, H. P. Rogers, Henry Schutzberger, J. L. Slechticky, O. L. Wadkins, Earl Walden, D. L. Webb, R. L. Westhafer, Charles Wexler.

At the business meeting the following officers were elected for the year 1951–1952: Chairman, Charles Wexler, Arizona State College, Tempe, Arizona; Vice-Chairman, J. H. Butchart, Arizona State College, Flagstaff, Arizona; Lecturer, two-year term, D. L. Webb, University of Arizona. The section accepted the invitation of the University of Arizona to hold the spring meeting, April 11–12, 1952 in Tucson, Arizona.

Friday evening, March 23, Dr. B. W. Jones of the University of Colorado delivered an address by invitation entitled *Mathematics from the Cradle to the Grave*. Dr. Jones pointed out numerous examples of related problems in mathematics that can be introduced into standard mathematics courses at all age levels.

The following papers were read in the two-day session:

1. *Radial heat flow in a hypersphere*, by Professor R. L. Westhafer, New Mexico College of A. & M.A.

Upon obtaining the partial differential equation of radial heat flow in a hypersphere by a method analogous to that used in a three dimensional sphere, it was shown that the temperature function has the same behavior as the angular velocity of a viscous, incompressible fluid in an infinite right circular cylinder rotating about its axis.

2. *A necessary and sufficient condition for the existence of a unique least squares solution*, by Professor M. S. Hendrickson, University of New Mexico.

A necessary and sufficient condition that there be a unique solution to the problem of find-

ing by the method of least squares a function of the form $y=\sum_{j=1}^{m} a_j f_j(x)$ to fit the $n$ points $(x_i, y_i)$, $n>m$ is that the matrix $\|f_i(x_j)\|$ $i=1, \cdots, m$ column index, $j=1, \cdots, n$, row index, have rank $m$.

3. *Unsolved problems of convex bodies*, by Dr. P. C. Hammer, University of California, Los Alamos, New Mexico.

Let $C$ be a closed convex body in a plane, and let $x$ and $y$ be any two points not on the boundary of $C$. The lengths of the chords lying on rays through $x$ and $y$ are determined. Is $C$ the only convex body having the same chord length functions? This class of problems including generalization to higher dimensions and allowing $x$ and $y$ to go to infinity in two specified directions, the author proposed about three years ago. At present it is known that for two perpendicular directions in the plane the chord lengths do not uniquely determine a convex body. It is known that adding one more direction will uniquely determine the body or that some angle between two lines other than $90^0$ will yield uniqueness.

The author defines an *outwardly simple* family of lines in the plane as a family of lines which simply cover the exterior of some circle including the points at infinity. Consider the convex hull $H$ of the intersection points of such a family. It is conjectured that the total accumulated area swept out by the lines inside $H$ is at most three times the area of $H$ and that it is three times the area of $H$ if and only if $H$ is a triangle and the lines are its extended diameters.

4. *Numbers represented by certain quadratic forms*, by Professor B. W. Jones, University of Colorado.

If $p$ is a prime congruent to 1 (mod 12), it is represented by one and only one of the forms $f=x^2+36y^2$, $g=9x^2+4y^2$. The background for this result was sketched as well as the basis for the fact that $-3$ is a biquadratic residue or non-residue of $p$ according as $p=g$ or $p=f$ is solvable.

5. *On the distribution of digits of powers*, Professor Leo Moser, Texas Technological College.

The following theorem was proved and generalized in several directions. Every finite sequence of digits appears as the first digits of some power 2. Related solved and unsolved problems on the normality of certain sequences were discussed. The first part of this paper, written with co-author N. Macon, will be published in *Scripta Mathematica*.

6. *Vector methods in the geometry of the tetrahedron*, by Professor F. C. Gentry, Arizona State College, Tempe, Arizona.

Vectors were used to determine the configuration of points, lines, planes, and spheres associated with the bisecting planes of the internal and external dihedral angles of a tetrahedron. The six Apollonian spheres were shown to be co-axial.

7. *A class of infinite groups*, by Professor D. G. Duncan, University of Arizona.

The class of groups whose elements are vectors of the form $[c, f(x)]$, where $f(x)$ is a polynomial whose coefficients along with $c$ belong to $GF(p)$ has been studied by L. Kaloujnine. The group operation $X$ being defined here by $[c, f(x)]X[d, f(x)]=[c+d, f(x)+g(x+c)]$. The class of groups arising when the coefficients belong to certain infinite sets has been investigated by the speaker. The special cases where coefficients belong (i) to the integers, (ii) to the rational number field, were presented. In particular the structure of the lower central and derived series was determined as well as the generating elements of the Abelian factor groups and of the subgroups appearing in the central series.

8. *Some properties of generators of a three-valued Post algebra*, by Professor D. L. Webb, University of Arizona.

Limitations were developed on binary generators of a three-valued Post algebra. In the case of Boolean algebra these conditions were sufficient to determine the Sheffer stroke function.

9. *Demonstration of a machine for playing the game Nim*, by Mr. Frank Kros, Assistant P. Cioffi, Student Harold Connell, New Mexico College of A. & M. A., introduced by the Secretary.

Mr. Kros explained the principles of the game Nim and Mr. Cioffi and Mr. Connell demonstrated the machine built by Mr. Connell under the supervision of Professor Harold Brown of the A. & M. electrical engineering department. The machine was constructed following the specifications, with slight modifications, of the article by Raymond Redheffer in the June 1948 issue of this MONTHLY.

10. *Formation of a simple irrational equation*, by Professor J. W. Beach, University of New Mexico.

An expression, in terms of possible roots, was given for each of two forms of irrational equations which reduce to quadratics. The possible roots were chosen arbitrarily and the conditions were given which would make both roots satisfy, both be extraneous, or one satisfy while the other would be extraneous.

11. *Reducing the general conic*, by Professor J. H. Butchart, Arizona State College, Flagstaff, Arizona.

The speaker feels that texts on analytic geometry should call attention to the matrix of coefficients of the general conic as a means of recalling the equations satisfied by the coordinates of the center and the expression for the constant after translating the origin to the center. Then the final coefficients of $x^2$ and $y^2$ are easily found from the expression for their sum and their difference. For the parabola, the rotation is first handled in the same manner, after which translation may be effected by completing a square.

12. *On the logic and teaching of equality*, by Professor R. S. Fouch, Arizona State College, Tempe, Arizona.

Almost all high school and college texts take the equality as a concept already understood; most introductory texts state a number (from four to ten or more) of postulates of equality and make tacit use of several more. The possibility of making simple and meaningful proofs of these properties is generally neglected. Hilbert and others proved the complete properties of equality from the two postulates: $a=a$ and $a=b\rightarrow[P(a)\rightarrow P(b)]$. It was shown that the same thing can be accomplished from either of two definitions: (1) "$a=b"=df$ "$P(a)\leftrightarrow P(b)$" or (2) "$a=b"=df$ ($\exists|x$) "$[aNx$ and $bNx]$", where "$aNx$" means that "$a$" is a name for $x$. It was argued that either or both of these definitions could be used in introductory courses and would produce pedagogic advantages such as greater clarity and a better foundation of basic concepts.

13. *Panel discussion: High school preparation for college mathematics*, by Earl Walden (Moderator), New Mexico College of A. & M. A., Charles Wexler, Arizona State College, Tempe, Max Kramer, New Mexico College of A. & M. A.

Two topics were discussed: *What should be the college entrance requirements in mathematics?* and *What can the colleges do to improve high school mathematics?* It was pointed out that by general cooperation colleges that have no entrance requirements in mathematics can and should insist on

two years of mathematics for entrance, provided that the selection be made from a broad base of mathematical subjects. The date for such change should be set sufficiently in advance to enable high schools to establish "two or three track" systems for their students.

Colleges can help high school mathematics programs by intensive training of prospective mathematics teachers in mathematics as well as education, deliberately stressing the relationship of advanced mathematics to the work of the secondary school, and by encouraging the most competent to teach. It was also argued that colleges provide intellectual stimulation for the gifted by visiting lecturers, awarding prizes, medals and scholarships, by sponsoring tournaments, clubs, and school publications. All students, however, should be provided with the guidance pamphlets recently issued by the M.A.A. and the N.C.T.M.

The panel of this discussion was appointed as a committee to meet with a similar committee of secondary school teachers to work on common problems and make recommendations to the Section at the 1952 meeting.

R. L. WESTHAFER, *Secretary*

## CALENDAR OF FUTURE MEETINGS

Thirty-fifth Annual Meeting, Brown University, Providence, Rhode Island, December 29, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Waynesburg College, Waynesburg, Pennsylvania, May 10, 1952.

ILLINOIS, Western Illinois State College, Macomb, May 9–10, 1952.

INDIANA, Indiana University, Bloomington, Spring, 1952.

IOWA, Coe College, Cedar Rapids, April 18–19, 1952.

KANSAS

KENTUCKY, University of Kentucky, Lexington.

LOUISIANA-MISSISSIPPI, Northwestern State College, Natchitoches, Louisiana, February 15–16, 1952.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, National Bureau of Standards, Washington, D. C., December 8, 1951.

METROPOLITAN NEW YORK, Spring, 1952.

MICHIGAN, University of Michigan, Ann Arbor, April 12, 1952.

MINNESOTA, North Dakota Agricultural College, Fargo, October 6, 1951.

MISSOURI, Lindenwood College, St. Charles, Spring, 1952.

NEBRASKA

NORTHERN CALIFORNIA, University of California, Berkeley, January 26, 1952.

OHIO, April 19, 1952.

OKLAHOMA, Oklahoma City University, October 12, 1951.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 20, 1952.

PHILADELPHIA, University of Pennsylvania, Philadelphia, November 24, 1951.

ROCKY MOUNTAIN, Western State College, Gunnison, Colorado, May, 1952.

SOUTHEASTERN, Georgia Institute of Technology and Agnes Scott College, Atlanta, March 21–22, 1952.

SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 8, 1952.

SOUTHWESTERN, University of Arizona, Tucson, April 11–12, 1952.

TEXAS, East Texas State Teachers College, Commerce, April, 1952.

UPPER NEW YORK STATE, Colleges of the Seneca, Geneva, May, 1952.

WISCONSIN

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